Knot Invariants

Math 1410 - Final Project
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Contents

1 Introduction

1.1 Motivation

Knots are everywhere. What, on surface level, perhaps seems the domain of sailors and scouts, is actually deeply woven into biology, physics, and history. In fact, we don't have to look very far in our everyday lives to find knots. Anyone who has lived within the last quarter-century has almost certainly experienced the frustration of pulling earbuds out of their pocket or a corded mouse from a box, only to find their wires tangled in some incomprehensible mess. In the subsequent minutes of untangling, a curious mind might wonder if they've ever seen this type of knot before. What does it mean for knots to be of the same *type*? What makes knots different?

Before we dive into the rabbit hole of knot theory, it should prove useful to prep ourselves with appropriate definitions.

Definition 1 *A knot is any embedding of* S^1 *in* $\mathbb{E}^3$.

That is, we'll say a knot is any subspace of three dimensions homeomorphic to the circle. Note this differs from our usual notion of a knot by the fact that our definition requires the ends of the curve to be fused together.

To understand when two knots are equivalent, we'll define the notion of an ambient isotopy.

Definition 2 *Two knots* k_1 *and* k_2 *are ambient isotopic if there exists a homotopy* $H : \mathbb{E}^3 \times I \to \mathbb{E}^3$ *such that each* H_t *is a homeomorphism,* H_0 *is the identity map, and* $H_1 \circ k_1 = k_2$.

In other words, if there exists an ambient isotopy between two knots, one can, in an intuitive sense, stretch and unravel one knot into the other without cutting or gluing. From here, unless otherwise stated, we'll say two knots are the same if there exists an ambient isotopy between them. So, when classifying knots and discussing types of knots, we're really talking about knots under the equivalence class of ambient isotopy.

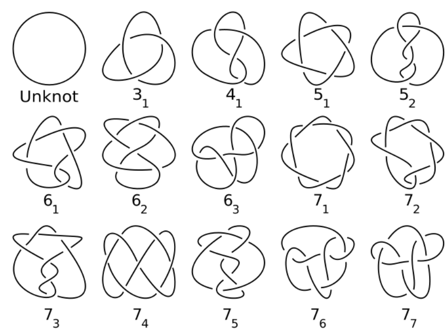
When discussing knots, visualizations best communicate the subtleties of each curve. Of course, we're limited by the two-dimensionality of paper, so it's worth taking a moment to formalize our diagrams. We call a knot *polygonal* if it is composed of a finite number of line-segments. In this paper, we shall restrict ourselves to those knots equivalent to polygonal knots[1]. We shall use so-called "nice" projections; that is, our projection diagram of a knot k is such that no more than two points of k are mapped to the same point in our projection, there are a finite number of these pairs of points of k, and no pair contains a vertex of k. The diagrams of these projections have *overpasses* interrupting *underpasses*. We will call each curve between these interrupts *arcs*[2]. Now, we're more equipped to dissect knots.

[1] Avoiding the infinitude of "wild" knots.

[2] It's not incredibly obvious that every knot has a "nice" projection. A quick proof-sketch can be found in (Armstrong, 215–216), but we'll take it for granted here.

1.2 Knot Table

The following is a table of prime[3] knots with seven minimum crossings and fewer.



Often, we'll refer to this table and call a certain knot C_a, where C is the number of crossings and a is the given virtually arbitrary ordering of nonequivalent knots.

So, how do we know that the figure-8 knot, 4_1, really isn't the unknot in disguise? How can we be certain any of these knots can't be cleverly pulled, twisted, or rearranged to create any others in this table?

1.3 Reidemeister Moves

In the 1920's, Kurt Reidemeister developed what is arguably one of the most useful notions in studying knot diagrams: Reidemeister Moves. He devised three different moves one could make to a knot diagram, shown in Figure 1.

Move *I*, sometimes called "twist", introduces a loop in one arc. Move *II*, or "poke", slides one arc over another. Note how one strand owns both overcrossings. Move *III*, "slide", slides an arc over a crossing. All of these moves can be reversed. It's clear, from an intuitive sense, that these moves don't change the knot. Twisting an unknot, for example, yields the same knot. So clearly, any finite sequence of Reidemeister Moves on a knot k yields a knot equivalent to k. What Reidemeister more brilliantly proved was that the converse is also true.

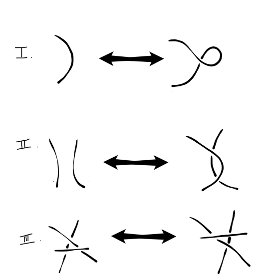


Figure 1: The Reidemeister Moves

[3]A prime knot, similar to a prime number, is one that is not equivalent to a "composition" of other knots. Consider cutting two trefoil knots, 3_1, and attaching the ends of one to the ends of the other. What's been composed here is a new knot with six crossings, but this knot doesn't reside in our table. It's not relevant to this paper, but an interesting exercise is finding which group-structure axiom this "composition" breaks on the equivalence set of knots.

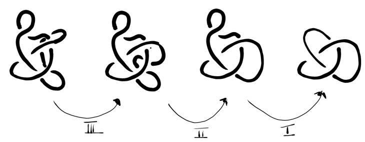


Figure 2: A series of Reidemeister Moves to show two knots are equivalent

Theorem 1 *(Reidemeister Theorem) Two knots are equivalent if and only if one of the knot's projections can obtain the other via a sequence of Reidemeister Moves and trivial manipulations of the projection.*

In other words, we have that if two knots are equivalent, we can find some finite sequence of Reidemeister Moves that turns the first knot into the second, not including trivial stretches and movements of the knot. Figure 3 shows an example of some knot being equal to the trefoil knot in only three of these moves.

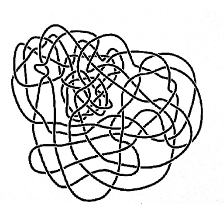


Figure 3: Wolfgang Haken's Gordian Knot

So, we now have some way of determining if two knots are equivalent without the need of deriving some isotopy between them! We can show knot equivalences, like the "Gordian knot" in Figure 1.3 actually being equivalent to the unknot[4]. Suppose for a second, though, that some knotty[5] person gave you a 5_2 and said, "This is actually a figure-8 knot (4_1). Show me the sequence of Reidemeister Moves to get there." And so you toil over it, but come up empty handed. You have no way of disproving his claim, which is indeed false. We can't try every finite sequence of Reidemeister Moves (of which there are infinite); what we need is a way to determine if two given knots are not equivalent.

2 Invariants

Our goal now is to find a way to show that two given knots are not equivalent. We will look for properties of knots that hold for some knots but not others – ideally, we want properties unique to a given equivalence class of knots. The key is, though, that these properties hold for all knots of the same type. We call these properties *knot invariants*.

[4]The proof is left as an exercise for the masochist.

[5]sorry

2.1 Colorability

We start by analyzing what happens when we color knots. Armed with a few crayons and a juice box, we define the following:

Definition 3 *A knot diagram is colorable if it can be colored in such a way that*

(1) Each arc is colored one of three colors,

(2) At least two colors are used, and

(3) At any crossing where there are two colors, there must be three colors.

Figure 4: The trefoil knot is colorable

Using this property, we can begin to classify different knots. Firstly, we can observe that the unknot is not colorable, since any attempt to do so either breaks rule 1 or rule 2. However, the trefoil *is* colorable (Figure 4). And as we color more knot diagrams (which I encourage the reader unfamiliar with this concept to do), we see that diagrams representing the same knot share the property of colorability. Figure 5 shows a few examples of this with the unknot. So, does this always hold true? The answer is yes. For any colorable knot k, every diagram of k is colorable.

Theorem 2 *Colorability is a knot invariant.*

Proof. By the Reidemeister Theorem, it suffices to show that the colorability property is consistent between Reidemeister Moves. Let k be some knot.

Move I. We can show that if a k is colorable, then adding a twist leaves it colorable. So, suppose k is colorable and we add a twist to some arc a with color c. What we are left with is k with two new arcs in the place of a, which we'll call a_1 and a_2. We can simply color a_1 and a_2 with c, keeping the rest of the diagram the same. The new crossing we made has only one color, which doesn't break any of our coloring rules. Thus, colorability is consistent over Move I.

Move II. We need to show that if two strands are colorable, then the diagram resulting from *Move II* is also colorable. Suppose k is colorable and a and b are two arcs. We have two cases:

Case 1: Suppose a and b are the same color c. Well, after we introduce a poke, we can color all new arcs c, keeping the rest of the diagram the same. Thus, we retain colorability.

Case 2: Suppose a is colored c_a and b is colored c_b. Let's zoom into these two arcs and label the resulting diagram such that b is broken up into three arcs, b_1, b_2, b_3, where b_1 and b_3 share the same endpoints as b. We can color a with c_a, and color b_1 and b_3 with c_b. For b_2, all we have to do is chose a new color, say C, and we can check that this arrangement doesn't break any rules.

What's left is analyzing Move III. We can do the same simple analysis by cases as above. Assuming you can convince yourself of this, we end the proof here. □

So, we come across our first distinction result: the trefoil knot is not equivalent to the unknot. But it doesn't take much crayon-work to figure out that the figure-8 knot is also not colorable. Unfortunately, we have a binary property – we can only distinguish between two buckets of knots, colorable and not[6]. We move on in search of more decisive invariants.

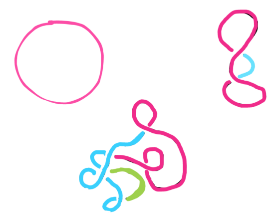


Figure 5: Different unknot diagrams fail the colorability test

2.2 Alexander Polynomials

Our goal in this section is to derive a polynomial for each knot in hopes of finding a more revealing property about differences in knots. First, we should expand our understanding of crossings.

We can give a knot some orientation by simply deciding a direction to run our finger along, which we will denote in our diagrams with arrows. This way, we can classify crossings in two ways.

Definition 4 *A right-hand crossing is one that aligns with one's right hand, where the palm is faced away from the knot, the thumb aligned with the overcrossing, and the remaining fingers with the undercrossing. A left-hand crossing is just that, but with the left hand.*

Now we want to define an $n \times n$ matrix where n is the number of crossings. We start by enumerating each arc, then enumerating each crossing separately. We made the distinction between left- and right- handed crossings so that we can make the following labelings, as in Figure 6:

Let a be the overcrossing arc, b be the undercrossing arc leaving the crossing, and c the undercrossing arc entering the crossing. Then, we'll enter each cell in the matrix by evaluating each crossing as follows:
If the crossing i is left-handed, the ath column of the ith row is $1-t$, the bth column of the ith row is -1, and the cth column of the ith row is t.
Otherwise, if the crossing i is right-handed, the ath column of the ith row is $1-t$, the bth column of that row is t, and the cth column of that row is -1.

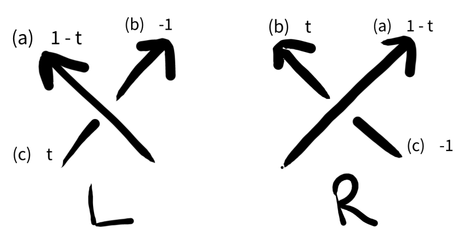


Figure 6: Left- and right-handed crossings, respectively

[6] There's actually an interesting generalization of coloring, *p-coloring*, which, for some prime number p, looks at colorability with p colors; see (Livingston, 39). We've gained enough inspiration from 3-coloring, though, to delve into more interesting knot invariants.

What we now have is a matrix of linear terms of t. Now, we eliminate the last row and last column of the matrix. Let M be the $(n-1)\times(n-1)$ matrix defined in this way.

Definition 5 *For some knot K, M as calculated above is the Alexander Matrix of K. The Alexander Polynomial is $A_K(t) = det(M)$.*

As an example, we shall calculate the Alexander Polynomial of the true lovers' knot[7] (Figure 7). We start by lettering each crossing and enumerating each arc. Note that the chosen orientation is such that each crossing forms a right-hand crossing. So, we go through each crossing and fill out our matrix as so:

$$
\begin{array}{c}
\\ \mathrm{a} \\ b \\ c \\ d \\ e \\ f \\ g \\ h
\end{array}
\begin{array}{c}
\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{pmatrix}
1-t & 0 & 0 & -1 & t & 0 & 0 & 0 \\
t & 0 & 0 & 0 & 1-t & 0 & 0 & -1 \\
1-t & 0 & 0 & 0 & 0 & -1 & t & 0 \\
-1 & t & 0 & 0 & 0 & 0 & 1-t & 0 \\
0 & 1-t & 0 & 0 & 0 & 0 & -1 & t \\
0 & -1 & t & 0 & 1-t & 0 & 0 & 0 \\
0 & 0 & 1-t & 0 & -1 & t & 0 & 0 \\
0 & 0 & -1 & t & 0 & 1-t & 0 & 0
\end{pmatrix}
\end{array}
$$

We cross out column 8 and row h, and proceed to take the determinate.

$$
(1-t)\begin{vmatrix}
0 & 0 & 0 & 1-t & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & t \\
t & 0 & 0 & 0 & 0 & 1-t \\
1-t & 0 & 0 & 0 & 0 & -1 \\
-1 & t & 0 & 1-t & 0 & 0 \\
0 & 1-t & 0 & -1 & t & 0
\end{vmatrix}
-(-1)\begin{vmatrix}
t & 0 & 0 & 1-t & 0 & 0 \\
1-t & 0 & 0 & 0 & -1 & t \\
-1 & t & 0 & 0 & 0 & 1-t \\
0 & 1-t & 0 & 0 & 0 & -1 \\
0 & -1 & t & 1-t & 0 & 0 \\
0 & 0 & 1-t & -1 & t & 0
\end{vmatrix}
$$

$$
+t\begin{vmatrix}
t & 0 & 0 & 0 & 0 & 0 \\
1-t & 0 & 0 & 0 & -1 & t \\
-1 & t & 0 & 0 & 0 & 1-t \\
0 & 1-t & 0 & 0 & 0 & -1 \\
0 & -1 & t & 0 & 0 & 0 \\
0 & 0 & 1-t & 0 & t & 0
\end{vmatrix}
$$

[7] This is Armstrong Chapter 10, Problem 30.

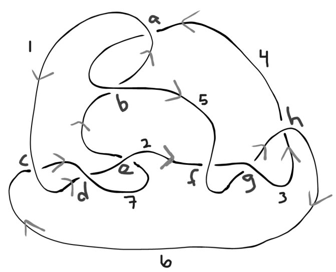


Figure 7: True Lover's Knot (8_{19}) with chosen orientation and labelings

The remainder of this calculation, which I will omit, leads to a final polynomial of the form $t^6 - t^5 + t^3 - t + 1$. The reader who reads this paper in the mirror, however, might calculate the polynomial after replacing each right-handed crossing with a left-handed crossing. And it's very much possible that this polynomial is different from the one calculated above. And, as it turns out, the same might happen if one introduces any number of Reidemeister moves to the diagram. Fear not! Closer observation yields the following result.

Theorem 3 *The Alexander Polynomial of a knot is a knot invariant, up to a multiple of $\pm t^k$, for some integer k.*

The proof for this is quite involved and takes more linear algebra than wanted, but, in essence, one needs to carefully show two things. The first is that performing a Reidemeister Move on a knot is consistent with the Alexander Polynomial; the second is analyzing a switch in orientation on the Alexander Polynomial. Because an Alexander Polynomial is consistent up to a multiple of t^k, texts often standardize the polynomial by reducing it as a Laurent polynomial (that is, $A_K(t) \in \mathbb{Z}[t, t^{-1}]$). Thus, the standard Alexander Polynomial for the true lovers' knot is $t^3 - t^2 + 1 - t^{-2} + t^{-3}$. For brevity, texts often shorten this polynomial to be just a list of coefficients. The following is a list[8] of Alexander Polynomials for the knots in the knot table (Section 1.2), where the number in brackets is the constant term in Laurent polynomial form:

3_1: 1 [-1] 1
4_1: -1 [3] -1
5_1: 1 -1 [1] -1 1
5_2: 2 [-3] 2
6_1: -2 [5] -2
6_2: -1 3 [-3] 3 -1
6_3: 1 -3 [5] -3 1
7_1: 1 -1 1 [-1] 1 -1 1
7_2: 3 [-5] 3
7_3: 2 -3 [3] -3 2
7_4: 4 [-7] 4
7_5: 2 -4 [5] -4 2
7_6: -1 5 [-7] 5 -1
7_7: 1 -5 [9] -5 1

A few incredible properties of this polynomial become immediately apparent. For starters the symmetry. A closer inspection also shows that $A_K(1) = \pm 1$. These two

[8] From http://stoimenov.net/stoimeno/homepage/ptab/

properties indeed hold true for any knot! What's even more shocking and relevant to our goal is the uniqueness of these polynomials. We now have an even stronger, calculable method of distinguishing two knots. But we should be careful. It's not apparent from the knots above, but it is possible for two knots to share an Alexander polynomial. In fact, we only have to go as far as nine crossings to witness this: 9_{28} and 9_{29} both share the polynomial 1 -5 12 [-15] 12 -5 1. So, the Alexander Polynomial is quite a strong method for knot distinction, but it's not fully comprehensive. It seems that perhaps our best bet for distinguishing any two given knots is not to rely solely on one invariant, but rather to build up a tool-belt of many so that when one fails, we can try others[9]. Of course, when given two equivalent knots, this means that we would have no way of conclusively determining whether they are the same. Still, this method gives us a good chance when given knots that are different. So, let's expand our tool-belt with one more invariant.

2.3 The Knot Group

One with even a small topology background might find it hard to make it this far without thinking about the fundamental group of a knot. To talk about the fundamental group of a knot, however, is quite uninteresting – any knot has a fundamental group isomorphic to the integers, the same as a circle. We can make things infinitely more interesting, though, by looking at the compliment of a knot k, $\mathbb{E}^3 - k$.

Definition 6 *For some knot k, the knot group of k is defined as* $\pi_1(\mathbb{E}^3 - k)$.

It doesn't take much reasoning to convince oneself that equivalent knots have homeomorphic compliments in $\mathbb{E}^3$. What is necessary now is to calculate these groups. To start, let's set a base point for loops high above our knot in $\mathbb{E}^3$, and give k some orientation. If n is the number of crossings in k, then we let $\alpha_1, ..., \alpha_n$ be loops which wrap once around each overcrossing in such a way that is consistent with orientation for each α_i; here, we'll say each loop respects a right-handed crossing. Let l_i be the element of $\pi_1(\mathbb{E}^3_+ - k)$ determined by α_i, where $\mathbb{E}^3_+$ is the subspace of $\mathbb{E}^3$ above the plane $z = 0$. We can state the following.

Theorem 4 *The knot group of k is the free group generated by elements* $l_1, ..., l_n$ *with relations* $r_1, ..., r_{n-1}$*, where n is the number of crossings of k.*

We'll go over the main arguments of the proof of this theorem, but purposefully leave out some details that might require more building-up. The argument is to build up our space from pieces of which we can find the fundamental group, and then find the fundamental group of the whole by applying van Kampen's theorem[10] on the pieces. Firstly, we need to rearrange our knot a little bit. We'll assume that our knot k has a nice projection on $z = 0$. We want to break up the arcs of k such that each underpass is covered by a segment. We can then project each segment down onto the plane $z = 0$, and send a vertical line perpendicular to the plane up from the endpoints of these segments (Figure 8). It should be clear that this 'new' knot is still equivalent to k.

[9] As the old saying goes, "Don't put all your eggs into one knot invariant!"

[10] This quite dense theorem is worth its own time, but we'll be taking it for granted. A proof and discussion can be found at (Armstrong, 138).

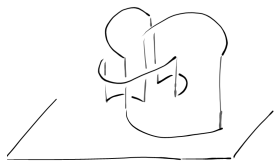


Figure 8: Trefoil with undercrossings projected to the plane $z = 0$

Let's call the overpasses plus the vertical lines $\hat{k}$; it's not hard to see that this is homotopic to k. We can then fill the space underneath the floating overpasses with a wall, and then thicken this wall carefully so that the thickened walls are still disjoint. We now have a set of disjoint balls in $\mathbb{E}^3_+$ which we can call $B_1, \ldots, B_n$. Imagine we hollow out each ball and remove the inner horseshoe shape from the plane $z = 0$. This new space is now homeomorphic to $\mathbb{E}^3_+$, thus it is simply connected. What we want to do now is construct $\mathbb{E}^3 - \hat{k}$ as the union of $X \cup (B_1 - \hat{k}) \cup \ldots \cup (B_n - \hat{k})$. Any of the balls without $\hat{k}$ is homeomorphic to a solid cylinder with a line from base to base cut out, which has a deformation retract to a disc with a hole. We know that this disc with a hole has a fundamental group isomorphic to the integers, and thus so does $(B_i - \hat{k})$. So now, suppose that $X \cup (B_1 - \hat{k}) \cup \ldots \cup (B_i - \hat{k})$ has a fundamental group that is the free group generated by $l_1, \ldots, l_i$ for some i. We want to add another wall $(B_{i+1} - \hat{k})$, but van Kampen's theorem tells us we need another generator, say l_{i+1}. Inductively, we now have that $\pi_1(\mathbb{E}^3_+ - k)$ is the free group generated by $l_1, \ldots, l_n$.

What's left is adding $\mathbb{E}^3_-$ to the mix. This takes a bit of algebra and is what introduces our relations $r_1, \ldots, r_{n-1}$, but otherwise runs somewhat similarly to the argument[11] for $\mathbb{E}^3_+$.

For our first example, we can look at the unknot. Well, we should already know that its knot group is isomorphic to the integers, but by the theorem above, we can conclude that it consists of one generator and no relations. Therefore, it is the infinite cyclic group. One can also imagine the knot group of the trefoil knot to be generated by some l_1, l_2, l_3, with relations as $l_1 l_2 = l_3 l_1$ and $l_2 l_3 = l_1 l_2$. Indeed, one can simplify this to be $\{a, b \,|\, aba = bab\}$.

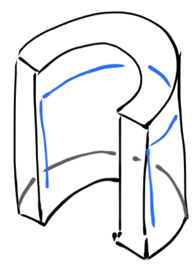


Figure 9: Some wall B_i, with $\hat{k}$ in blue

Simply listing off knot groups is almost certainly unsatisfying, so let's calculate one. We'll look at the true lovers' knot once again[12] – after all, what could be more romantic than calculating fundamental groups? We start by assigning to each arc a loop, respecting the right-hand rule, and labeling each a through h (Figure 10).

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[11] I refer the interested or unconvinced reader to (Armstrong, 220).

[12] This is Armstrong Chapter 10 Problem 3.

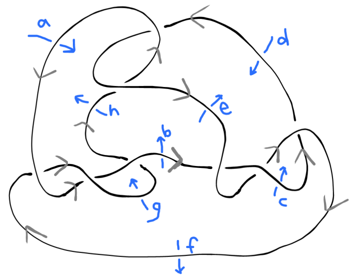

Figure 10: True lovers' knot, with $l_1,\ldots,l_8$ in blue

We know we're expecting eight generators and seven relations. We can start looking at each crossing. Let's zoom in on the crossing whose overpass is looped by a and underpasses are looped by f and g (Figure 11). We can see that the loop $f.a$ is the same as the loop $a.g$. Thus, we begin our set of relations by claiming $fa = ag$. Continuing this process, we get the following: the knot group of the true lovers' knot can be presented as

$$\langle a,b,c,d,e,f,g,h : fa = ag, ag = gb, gb = bh, he = ea, ae = da, be = ec, ec = fd \rangle.$$

And that's it! We have the tools necessary for calculating this invariant. As expected, multiple knots can share the same fundamental group presentation – on top of this, determining whether two of these presentations are isomorphic isn't a simple task. We concede the utility of this invariant in pursuit of its aesthetic. The knot group may not be the first tool one might want to reach for when first disambiguating two knots, but it sure is pretty. And here, we retire our search for knot invariants.

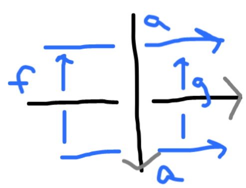

Figure 11: The leftmost crossing of Figure 10, showing $fa = ag$

3 Conclusion

We set out with a simple goal in mind: how do we distinguish between two knots? We found ourselves jumping through the domain of linear algebra, topology, and abstract algebra to find the treasures known as invariants. We're now equipped with three calculable invariants, both in procedure and in theory. When wanting to distinguish between two knots, we can reach for colorability, the simplest but weakest invariant we have. We can go for the Alexander Polynomial, which is strong and fairly simple to compute, but tedious the larger the number of crossings. Lastly, we can use the knot group of the knot, which is impractical but fun and clean. More important than the tool-belt we've developed, however, is that we've dipped our toes into a basin of knots that spans so far in any direction – we haven't even mentioned links, braid theory, knot-bounded surfaces, surgery, virtual knots, and so much more. Not to mention we've only discussed three knot invariants out of the many that exist. In short, knot theory is far more complex, far more in-depth, and has far more amazing ideas than what we've shown here.

Appendix

In studying knots, I've come across many interesting things, some not directly relevant to knot invariants, some not even mathematical. I respect the reader who is solely interested in knot invariants, and claim that the reader whose goal is only to grade this paper may feel free to conclude their reading here. But, I encourage those who, if anything like me, found the study of knots interesting to read on in hopes that I may spark a little more interest in the subject, of which weeks prior to writing I had virtually no knowledge of.

A: Braids

Braided hair, woven baskets, knitted textiles – we should be familiar with the intuitive notion of a braid, just as we were with knots. Braids share an interesting connection with knots. A braid can be defined as n number of curves with horizontally-aligned and fixed endpoints above and below, with the added rule that a curve should never go back upwards. From this definition, we can make a braid into a knot by drawing a line from each endpoint around the braid to the endpoint opposite it. Well, actually, we should be careful, since it's entirely possible to make a link composed of several pieces. Can we perform the reverse? That is, is there a way to take a knot and turn it into a braid? Yes! Take a trefoil, for example, and extend a radius from a point in the middle of its diagram outwards. So, the radius is always intersecting the knot in two places. Now, like radar, scan the radius around the knot. As the radius scans the knot, cross a braid as the radius passes over a crossing in such a way that keeps strands and arcs consistent. Once the radius scans a full 2π radians, the braid is complete, and we can fix the endpoints. The next question we might ask is, does this work for every braid? What J. W. H. Alexander, the namesake of the polynomial above, proved was that yes, it does indeed work for every knot! It's not so easy, though. If we try the same tactic with the figure-8 knot, our braid climbs back upwards before falling to the bottom endpoints. What it takes is some clever rearrangements of the arcs. So, we can say that for any knot diagram, there exists some finite sequence of Reidemeister Moves that allows the knot to construct a braid by the method above.

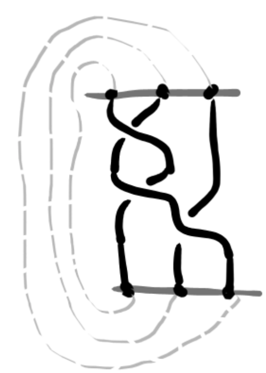

Figure 10: A braid with three strands, where the grey dotted line represents closing the braid into a knot

The last interesting thing about braids that I'll discuss here is the braid group. We can define a composition of braids as stacking two braids on top of one another, identifying the upper endpoints of one braid with the lower endpoints of another. It's not too difficult to show that this composition of braids has a nonabelian group structure. So, we can encode braids as algebraic words. We can call the move b_i the action of crossing the ith strand over the $i+1$th strand. Then, b_i^{-1} is the action of crossing the ith strand under the $i+1$th strand. So, we can say the braid in Figure 10 is encoded as $b_1 b_1 b_2 = b_1^2 b_2$. With this, we can show things like turning that braid into the identity consists of following the original braid by the braiding $b_2^{-1} b_1^{-2}$. Further, the nonabelian property of this group is only partially true. There exists commutativity for distant braids, that is

$$b_i b_j = b_j b_i \text{ when } |i-j| \geq 2, \quad i,j = 1,\ldots,n-1.$$

Another relation can also be worked out, named Artin's relation:

$$b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, \quad i = 1,\ldots,n-2.$$

These two relations let us show equivalences between complex braids. The following example is from (Sossinsky, 33):
Let a, A, b, B, c, C be the braids $b_1, b_1^{-1}, \ldots, b_3^{-1}$. Then

$$ABBAAAAAAAbbbbbbbbcbaAccBCaBBBBBBaaaaaaBB = e,$$

where e is the trivial braid. It's like a braid version of the Gordian Knot! So, a question to end on is: can these notions of braids help classify knots?

B: Biology and Knot Arithmetic

We are made up by knots. Knots undoubtedly play a role in DNA structure – there's a natural sense of braiding in DNA's molecular structure. DNA also has to coil up to fit itself into the nucleus of a cell. That by itself is pretty interesting, but my favorite piece of knotted biology has to do with a certain species of hagfish, the *myxine glutinosa*. This long eel-like sea-dweller has, in an evolutionary sense, a pathetic body. It has a distinct lack of jaw structure, making it difficult to eat prey, and it has soft, weak skin, making it easy to become prey. To compensate, this hagfish has a few tricks up its sleeve. It's able to move over itself, like a ribbon on the seafloor, until it's knotted (only needing to bite its tail to conform to our definition of a knot)! This allows it to spread mucus over its body so that it can slip and slide through, say, some shark's teeth. Knotting itself in this way also allows it to eat prey more easily. Interestingly, it's able to slide this knot up its body, so that even if a predator's grip withstood its slippery mucus, the hagfish can push against the grip with the knot! Here is a video of that in action: https://www.youtube.com/watch?v=RrPvMMkQkk0.

This action of sliding the 'knotted part' up the knot helps us understand something about knot arithmetic. This was discussed a little in footnote 3, but suppose we want to define an addition of knots. One way to do so is to say we can only construct knots inside some boxed region of $\mathbb{E}^3$ by taking some curve, knotting it up without fusing the ends together, and placing the endpoints on the midpoints of opposite faces of the box. Then, we have sort of building blocks, where we can take some amount of blocks and place them next to each other such that the identification of two faces implies the identification of two endpoints. We then connect the remaining two endpoints with a curve to close the knot. What properties does this addition hold? I'll leave that for the reader,

but spoiler – it doesn't induce a group structure on the set of knots. Nevertheless, it is commutative. Showing this is a bit mathematically intense, but for an intuitive sense, we can call upon our new friend, the hagfish! As the hagfish pushes the 'knotted part' of its knot up its body, we can push one knot up the the curve through some other knot. What a hideous creature to teach us about a beautiful token of math!
For more on knot arithmetic, see (Sossinsky, 46).

C: History

Knots have been a part of human culture for as long as humans have had materials to knot. From binding tools to religious practice to anchoring ships to weaving baskets, the knot had been too long neglected by the mathematician. It wasn't until 1860 that knots made their debut in the field of math, when William Thomson (also known as Lord Kelvin) had an epiphany. To set the stage, we're in a pre-Bohr era – the view of the atom is split into to camps. On one side, we have the corpusculars – the ones who believe atoms are single particles positioned in a precise point in space; on the other side, we have the wave theorists – those who believe atoms are superpositions of waves. In the middle sits Thomson, who one day proposes that an atom is neither a wave nor a particle, but rather a knot in the ambient ether! It was a marvelous idea to the scientific community at the time, and was taken quite seriously. Each element, he proposed, was thus a different knot. The unknot, the most basic of all knots, perhaps correlates to hydrogen; the trefoil with three crossings may correspond to lithium. So the goal shifted from something physiochemical to mathematical: if they were able to classify all knots, they could classify each element.

Thomson passed the work of this classification onto his friend Peter Guthrie Tait, who then devoted the rest of his life to tabulating knots. He went on to determine the first table of knots, which depicted each unique alternating knot up to ten crossings (alternating meaning he restricted his study of knots to only those whose crossings alternated between over- and undercrossings). Unfortunately, by the time he finished, Mendeleev had uprooted Thomson's original idea with his new periodic table of elements...a tragic ending to Tait's life's work, which is now appreciated by the many knot theorists who reference his tables.
For more, see (Sossinsky, 1).

Sources

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