Classification of Groups by the Genus of Cayley Graphs

A Graph Theoretic Survey of the Study of Group Genera

Math 1230 - Final Project
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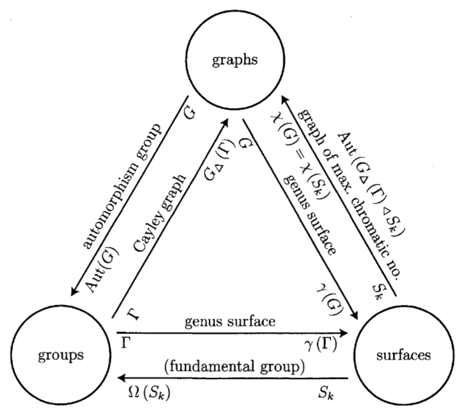
Contents

1 Motivation

A graph, a group, and a surface walk into a bar

These three structures are of course very different. With shallow thought, it's difficult to conjure up notable similarities or relations between all three. Between any two of these structures, it's possible to think of one in terms of the other: surfaces in terms of their fundamental groups, graphs in terms of their automorphism groups, etc. But mingling all three is a bit trickier.

Combining these relationships yields a method of observing properties of one structure *through* another structure, realized as a third structure. Of particular interest is the idea of realizing groups as graphs, and observing properties of that graph through surface embeddings. In other words, our goal is to take some Cayley graph $C(\Gamma)$ of some group Γ and classify Γ based on the genus of $C(\Gamma)$.



We'll set out to show three results: a theorem due to Levinson on the genus of infinite groups; a result by Tucker that shows any genus $\gamma > 1$ has only a finite number of associated groups; and finally a surprising result on the number of groups of genus 2. We'll end with a program I wrote that generates nonplanar Cayley graphs.

2 Basic Definitions

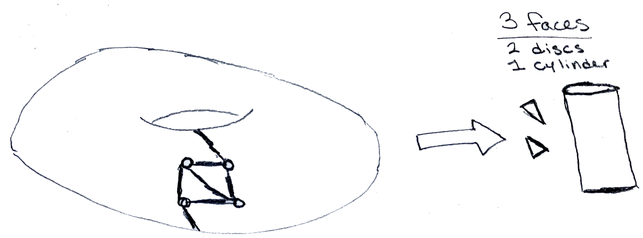
Graph Embeddings and Graph Genera

We'll start by building up some technology for understanding the graph-to-surface translation. Recall that we want to classify graphs based on the genus of the surface(s) we can embed our graph into. Here, we'll concern ourselves only with orientable surfaces.

Definition 1. *The* ***genus*** $\gamma(G)$ *of some graph G is the minimum genus of all surfaces that G can be embedded into. A graph embedding G in S_k (surface of genus k) is a* ***minimal embedding*** *if* $\gamma(G) = k$.

Definition 2. *A graph is called a* ***cellular embedding*** *if each face of the embedding is homeomorphic to the open disc.*

For example, any planar graph embedded in the sphere is cellular. But consider the graph K_4 embedded in the torus as shown.



We can see that the outer face is homeomorphic to the cylinder, so the embedding is not cellular. This is useful in narrowing our classification down, as we can develop a sense of an upper bound of a genus.

Definition 3. *The **maximum genus** $\gamma_M(G)$ of a connected graph G is the maximum genus among the genera of all surfaces in which G has a cellular embedding.*

The Genus of a Group

Recall that a Cayley graph of a group Γ with generating set S is a directed colored graph G whose vertices are the elements of Γ and which has an edge connecting two vertices v_1, v_2 if and only if g_1, g_2 corresponding to v_1, v_2 implies that there is some $s \in S$ such that $g_1 s = g_2$. Let $C_S(\Gamma)$ denote the undirected, uncolored underlying graph of the Cayley graph of Γ with generating set S. We'll use the term Cayley graph sometimes to refer to this underlying graph, since we don't care much about the directedness or coloring. By using the tools we've developed concerning the genus of graphs on the Cayley graph, we can develop an understanding of the "surface behavior" of groups.

Definition 4. *The **genus of a group** $\gamma(\Gamma)$ is the minimum genus of all Cayley graphs of Γ, i.e.*

$$\gamma(\Gamma) = \min\{\gamma(C_S(\Gamma))\}$$

taken across all generating sets S.

This definition comes from Arthur T. White, who, in the 70's, coined the notion of the genus of a group; yet, classification of groups by their genus can be dated as far back as 1896, when Mashke gave the following result for planar groups:

Theorem 2.1. *For a finite group Γ, $\gamma(\Gamma) = 0$ if and only if $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1 = \mathbb{Z}_1$ or $\mathbb{Z}_2$ and $\Gamma_2 = \mathbb{Z}_n, D_n, S_4, A_4,$ or A_5.*

3 Infinite Groups and Levinson's Result

Before going deeper into the classification of finite groups by their genus, it might first be natural to wonder what happens when we take groups of extreme order – how can we classify infinite groups? Consider the basic word group generated by $< a, b >$ and their inverses. The Cayley graph is the classic cauliflower design, one that's clearly planar. We can also consider an infinitely generated group whose Cayley graph contains K_n for any $n > 1$. By results similar to Kuratowski's Theorem, we'd eventually need greater and greater genera for this graph.

Definition 5. *An infinite graph G has **infinite genus** if for every $n > 0$, there exists a finite subgraph G_n of G such that $\gamma(G_n) \geq n$.*

We might wonder whether we can create infinite toroidal Cayley graphs or infinite Cayley graphs of any other genus. We'll need a couple of lemmas to build up to this answer.

Firstly, we'll want to understand how the genus of a graph corresponds to the sum of its components. We can show that the genus of a graph is the sum of the genera of its components; in fact, we can show the stronger statement that the genus of a graph is the sum of the genera of its blocks.

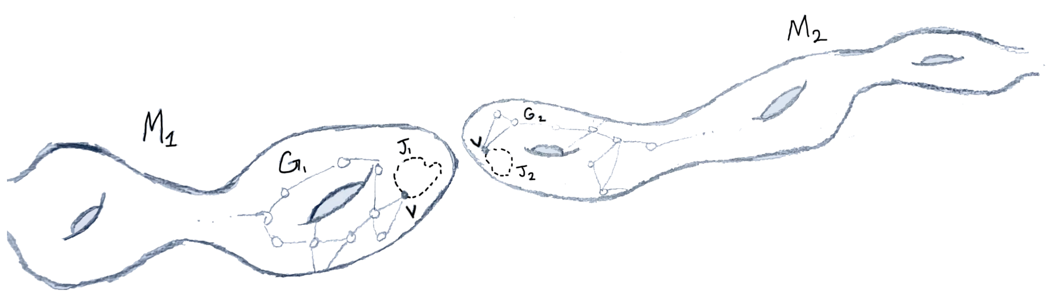
Definition 6. *A **block** of a graph is a maximally connected subgraph containing no cut-vertices.*

The proof of this additivity is done in (Battle) by showing $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$ and then $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$ for blocks G_1, G_2 such that $G_1 \cup G_2 = G$. The latter takes a bit of intense point-set topology, so we'll prove only the former with what I'll call the donut kissing method and defer the proof of the other inequality.

Lemma 3.1. *Let G_1, G_2 be blocks of G such that $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = v$. Then,*

$$\gamma(G) = \gamma(G_1) + \gamma(G_2).$$

Proof. We'll first show that $\gamma(G) \leq \gamma(G_1) + \gamma(G_2)$. Let G_1 and G_2 be minimally embedded into surfaces M_1 and M_2. We'll say v_i is v in G_i. Around v_i, we can draw a closed Jordan curve J_i in M_i such that the interior C_i is homeomorphic to the open disc and $(J_i \cup C_i) \cap G_i = v_i$.



Now, we can identify J_1 with J_2 such that $v_1 = v_2$, and get rid of C_i. What we're left with is a new surface of genus $\gamma(M_1) + \gamma(M_2)$ with G embedded into it. So, $\gamma(G) \leq \gamma(M_1) + \gamma(M_2)$.

We now want to show that $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$. For the proof of this, refer to (Battle).

Because of the inequalities shown above, $\gamma(G) = \gamma(G_2) + \gamma(G_2)$. □

If G is not connected, then we can consider the graph H constructed by taking all of the components of G, adding a vertex, and connecting that vertex to each component of G. Then we can apply basic induction to the lemma to get the following corollary.

Cor 3.2. *If G is a graph and $G_1, G_2, \ldots, G_n$ are the components of G, then*

$$\gamma(G) = \sum_{i=1}^{n} (\gamma(G_i)).$$

Lemma 3.3. *Let $G := C_S(\Gamma)$ for some infinite group Γ with generating set S. Let H be a finite subgraph of G. Then there exist two disjoint isomorphic copies of H in G.*

Proof. Because H is a finite subgraph of G, H corresponds to a finite set of elements of Γ, $\{g_1, g_2, \ldots, g_n\}$. Let

$$T = \{g_i g_j^{-1} \mid 1 \leq i, j \leq n\}.$$

Take some $x \in \Gamma - T$. Then, let

$$H' = \{x g_i \mid 1 \leq i \leq n\}.$$

H' is a subgraph of G by the closure of Γ. We'll show that H is isomorphic and disjoint to H'.
We can see that for $s \in S$, $g_i s = g_j$ if and only if $x g_i s = x g_j$, so the two sets are isomorphic. Suppose they weren't disjoint, so there exists some $v \in V(H) \cap V(H')$. Then there would be some i and j such that $g_i = x g_j$. That means that $x = g_i g_j^{-1}$, which contradicts that $x \in \Gamma - T$. So, H and H' are disjoint. □

Using these two lemmas, we can now prove the surprising Levinson's Result, classifying all groups of infinite order.

Theorem 3.4 (Levinson)**.** *If Γ is an infinite group, $\gamma(\Gamma) = 0$ or ∞.*

Proof. We gave an example of a planar group above. Suppose $\gamma(\Gamma) \neq 0$. Then there must exist some Kuratowski graph K (either K_5 or $K_{3,3}$, or some subdivision thereof) as a subgraph of $G := C_S(\Gamma)$. By 3.3, there must exist a K' in G that is a disjoint copy of K. By 3.2, we can see that $\gamma(G) \geq \gamma(K \cup K')$. Now we can apply 3.3 again to $K \cup K'$ to get 4 copies of K, and see that $\gamma(G) \geq \gamma(4K) = 4$. Continuing this, we get that for any n, we can find 2^n disjoint copies of K in G. Thus $\Gamma(G) = \infty$. □

4 Groups of Genus 1

After seeing all groups of genus 0, we might wonder how the list of groups of a given genus acts as we increase the genus. We'll look at some results by Proulx on genus 1 groups, but the proofs she lays out are far too intense or have way too many cases to be of interest here. Firstly, we'll state the following theorem that narrows down which groups can be toroidal.

Theorem 4.1. *If G is a toroidal group, then it has a presentation*

$$\langle x_1, x_2, \ldots, x_n : r_1 = r_2 = \cdots = r_s = 1 \rangle$$

such that if a is the number of elements in the generators $x_1, x_2, \ldots, x_n$ *with order 2, b is the number of generators with order 3, and c is the number of generators with order greater than 3, then*

$$3 \leq a + 5b/3 + 2c \leq 4.$$

Running through the cases of this inequality, we can see that a toroidal group must be generated by 2, 3, or 4 generators. I've written out all possible toroidal groups here. Taking the toroidal group of the form $\mathbb{Z}_n \times \mathbb{Z}_m$ for $\gcd(n,m) \geq 3$, for example, we can see that there are infinitely many toroidal groups.

If we wanted to take any surface with genus greater than 1, and, if we wanted to climb atop a high mountain and shout down "Hoorah! I know for certain that there are infinitely many Cayley graphs that I can embed into this surface!" (and honestly, who hasn't wanted to do this?), then we'd probably be kind of sad. Because, in fact, there is no genus greater than 1 for which we have infinite such graphs.

5 Groups of Genus ≥ 3

In 1980, Thomas Tucker published a paper on the number of groups of a given genus. He stated a very important theorem, the proof of which is too long to include here.

Theorem 5.1. *For any group* Γ*, if* $\gamma(\Gamma) > 1$*, then* $\gamma(\Gamma) \geq |\Gamma|/168 + 1$*. In particular, the number of groups of a given genus greater than 1 is finite.*

This is quite surprising, as, unlike groups of genus 0 or 1, we're limited in our selection of Cayley graphs. Let $\omega(\gamma)$ be the number of groups of genus γ. It is conjectured that for all $\gamma > 1$, $\omega(\gamma) \neq 0$.

There have been quite a few calculations for exact values of group genera, and some upper bounds have been installed. Some basic groups have surprisingly high genus. The group $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, for example, has a genus of 7. There are many strategies and techniques for calculating group genera, but it's surprisingly difficult. The example above gained attention in 1977, but wasn't solved until 1985.

6 Group of Genus 2

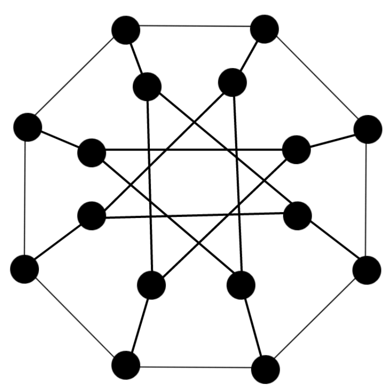
It would be appropriate to reread the section header, imagining an angelic choir singing in the background, because this is one of those rare and interesting tokenistic objects that shine in their uniqueness. Proulx conjectured in 1978 that there were no groups of genus 2. A few years later, Tucker published a complete classification of genus 2 groups. The classification can be stated as follows.

Theorem 6.1. *There is one group of genus 2. It is presented as*

$$\langle x, y, z : x^2 = y^2 = z^2 = 1, (xy)^2 = (yz)^3 = (xz)^8 = 1, y(xz)^4 y(xz)^4 = 1 \rangle.$$

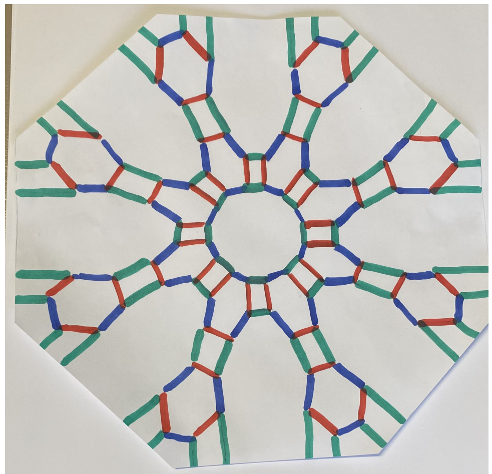
The proof of this is an exhaustive combinatorial checklist of different generator types combined with some group theory trickery; see (Tucker).

We'll call this group the Tucker group, or T for short. The group, which has 96 elements, is the automorphism group of the generalized Petersen graph $G(8,3)$, i.e., T is the set of symmetries[1] of the following graph:



The following is the Cayley graph of T:

[1] I've uploaded animations of the generators of these symmetries here.



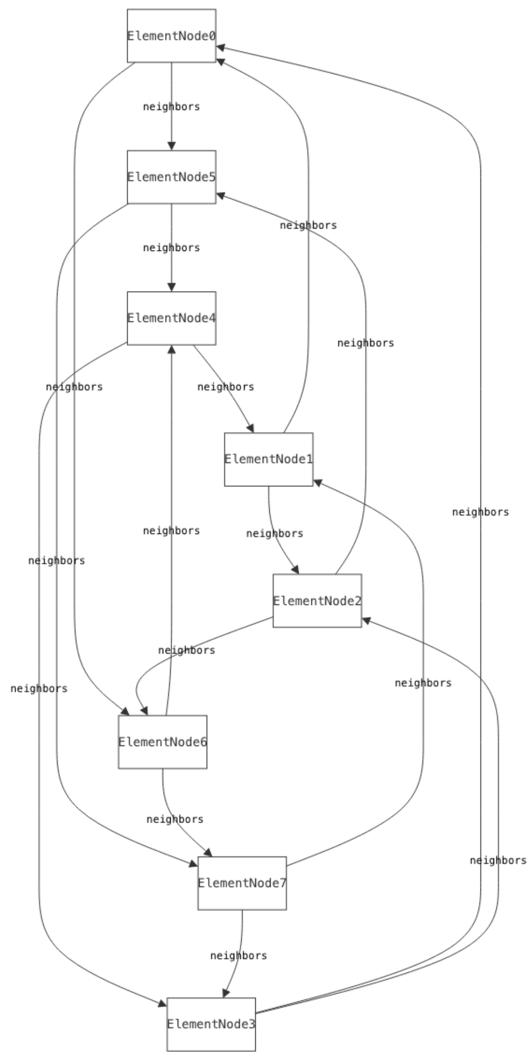
It is drawn on an octagon so that identifying opposite sides (without reversal) yields a surface of genus 2. The embedding is a symmetric embedding, with eight-fold rotational symmetry, which is 3-regular and 2-colorable. Not only does it look pretty cool, but it's amazingly symmetric.

7 Generating Non-planar Groups

In this section, we'll look at the result of code written by myself, Yizhong Hu, and Jordan Walendom for CS1710 (Logic for Systems). Our goal was to model many aspects of group theory in Forge so that we could exhaustively check theorems for small-ordered groups. By the vastness of group theory, we branched out in many directions, one of which was the application of graph theory. The code is here.

In generators.frg, we define minimal generators of groups, graphs, and subsequently Cayley graphs. The way it's set up, we're able to find instances of, for example, non-planar groups. A couple examples of results our code came to was:

- The smallest-ordered group of genus 1 is the quaternion group, which has order 8.
- Let the **potential genus** $\rho(\Gamma)$ be the *maximum* genus of all Cayley graphs of Γ with minimal generating sets. The smallest-ordered group with potential genus $\rho(\Gamma) = 1$ is again the quaternion group.



Forge-generated Visualization of Quaternion Group

The code is a lot of fun to play around with to generate Cayley graphs of small-ordered groups (Forge can only handle groups of around 12 elements or less, sadly).

8 Conclusion

We've seen that by taking a group and finding its Cayley graph with minimal genus, we're able to classify the groups. For groups of infinite order, we can surprisingly classify them as either genus 0 or genus infinity. For groups of order 0, we can classify pretty easily. For groups of order 1, it's much more difficult – there are 28 cases the group can take on. There are infinitely many groups of order 0 and 1, which makes those two genera unique for finite groups. Incredibly, there is exactly one group of genus 2. For genus greater than 2, we know the number of groups of a given genus is bounded above, and it's conjectured that it's bounded below by 1.

There is still a lot more to know about these groups. We haven't covered the strategies for calculating these genera – one of great ubiquity is calculating the rotation schemes of the graph – embeddings into non-orientable surfaces, the Euler characteristic of a group, etc. There are plenty of unanswered questions and plenty of room for greater calculations. In all, the topology of the graphs of groups spans greatly, and this paper has only scratched the surface.

Appendix A

Links

Here is a list of links I embed in the text, written out in case the hyperlinks are inaccessible:

- Animations of Symmetries of $G(8,3)$: https://beanway.me/projects/mobiuskantor.html
- All Toroidal Groups: https://beanway.me/projects/toroidalgroups.html
- Code from Section 7: https://github.com/YizhongHu/final_project/blob/master/generators.frg

9 Sources

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Gross, Jonathan L., and Thomas W. Tucker. Topological Graph Theory. Courier Corporation, 1987.

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"Section 10.6. Surface Embeddings of Graphs - East Tennessee State ..." East Tennessee State University, https://faculty.etsu.edu/gardnerr/5340/notes-Bondy-Murty-GT/Bondy-Murty-GT-10-6.pdf.

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