Searching the Geometric Bistellar Flip Graph

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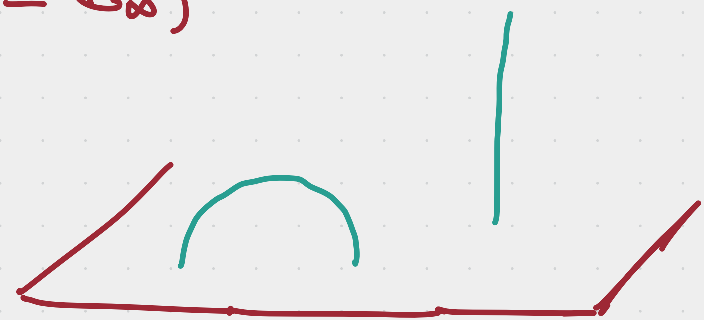
Let M be a cusped hyperbolic 3-manifold.

$$\rho : \pi_1(M) \hookrightarrow PSL_2(\mathbb{C})$$

such that $\rho(\pi_1(M)) \curvearrowright \mathbb{H}^3$ properly discontinuously (by Möbius transformations on $\partial\overline{\mathbb{H}^3} = \mathbb{C}_\infty$)



$\mathbb{H}^3 = \mathbb{C} \times \mathbb{R}_+$

$\mathbb{H}^3$

S^3 Sol

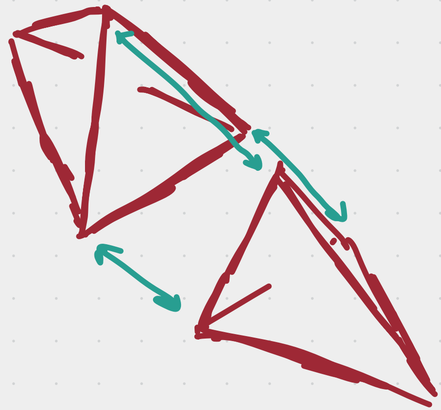
$\mathbb{E}^3$ Nil

$S^2 \times \mathbb{R}$ $\widetilde{SL_2\mathbb{R}}$

$\mathbb{H}^2 \times \mathbb{R}$

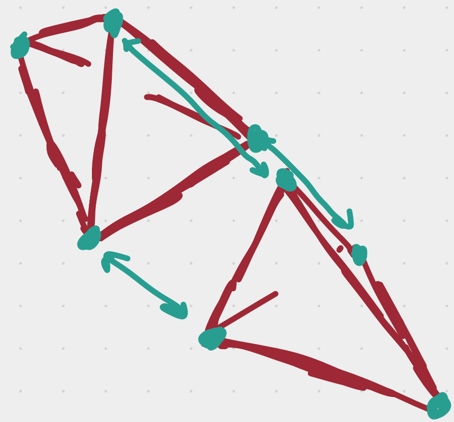
Hyperbolic is the most common yet least well-understood.

Thm (Moise) Every 3-manifold has a triangulation.



* Not true for 4-manifolds.

Thm (Moise) Every 3-manifold with ∂ has an ideal triangulation.

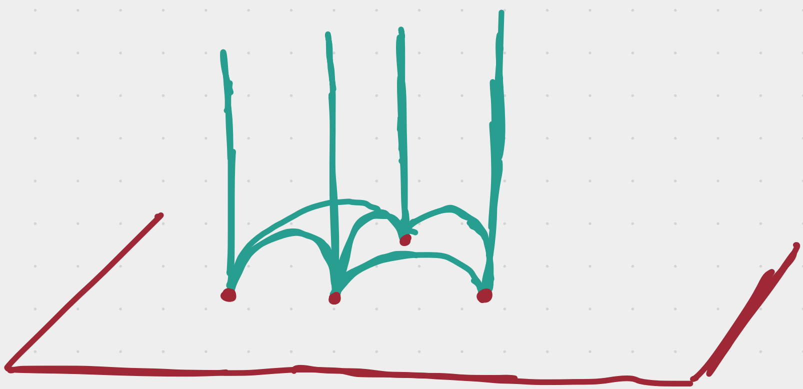


vertices $\leq 2M$

* Not true for 4-manifolds.

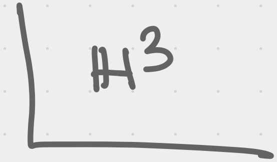
Thurston: Some triangulations can capture the geometry of M!

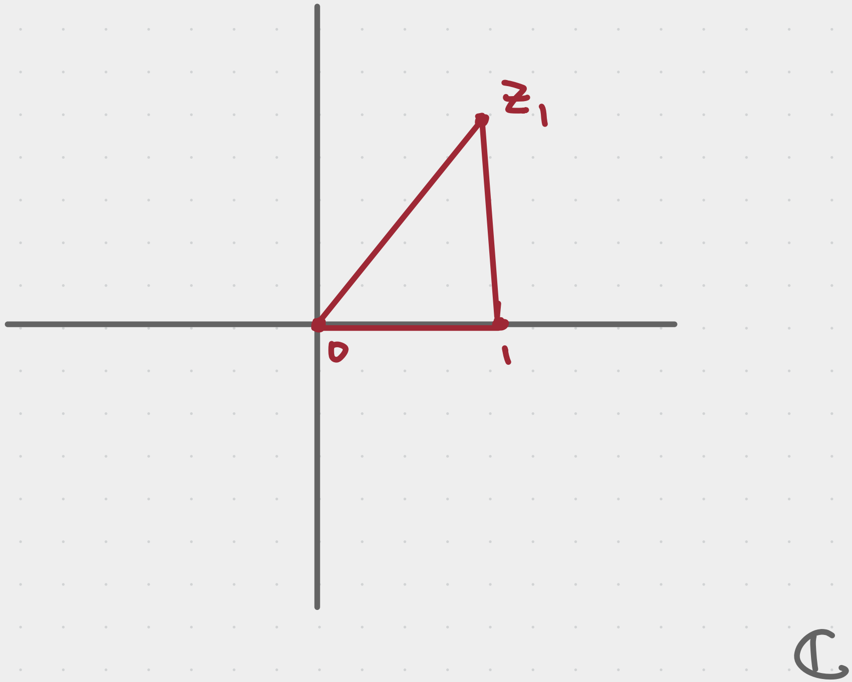
To check that a triangulation represents a triangulation by convex tetrahedra in M, we need to check every point has a neighborhood isometric to a ball in $\mathbb{H}^3$...



vertices $\subseteq \partial\mathbb{H}^3$

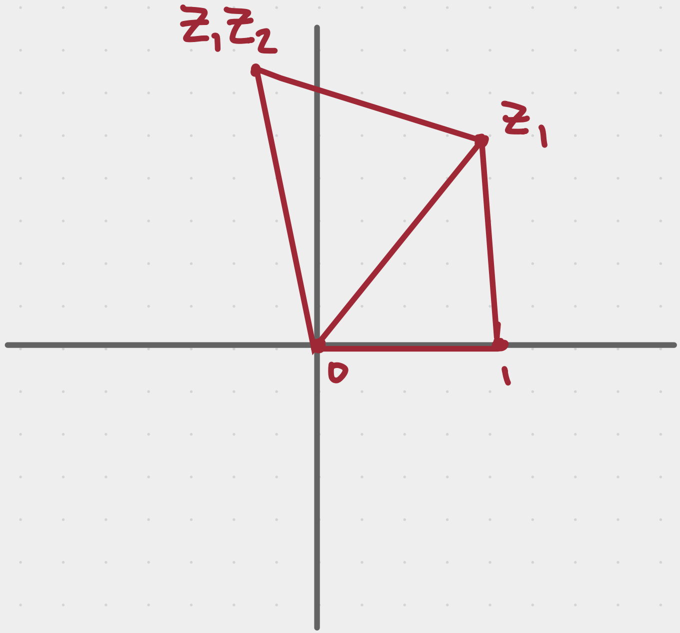
Pov ∞





POV ∞

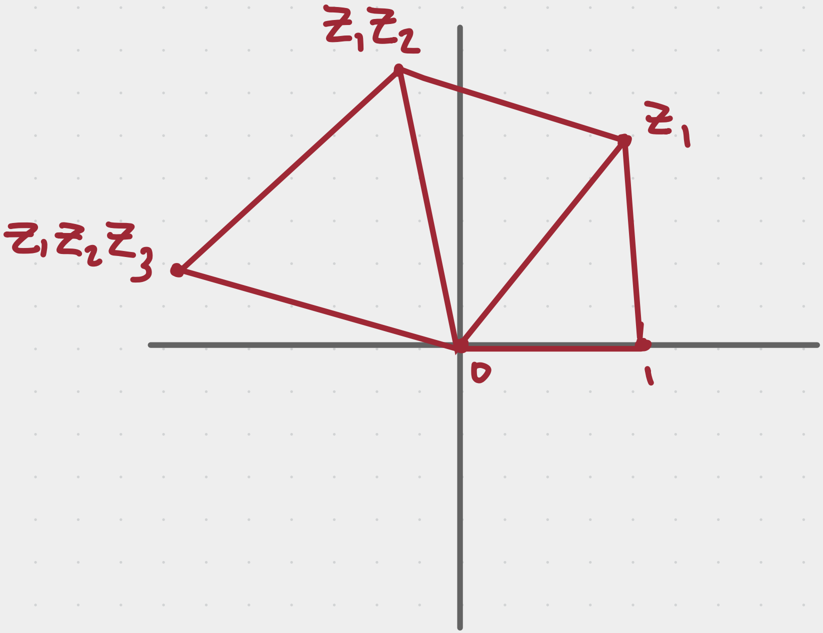
$\mathbb{H}^3$



$\mathbb{C}$

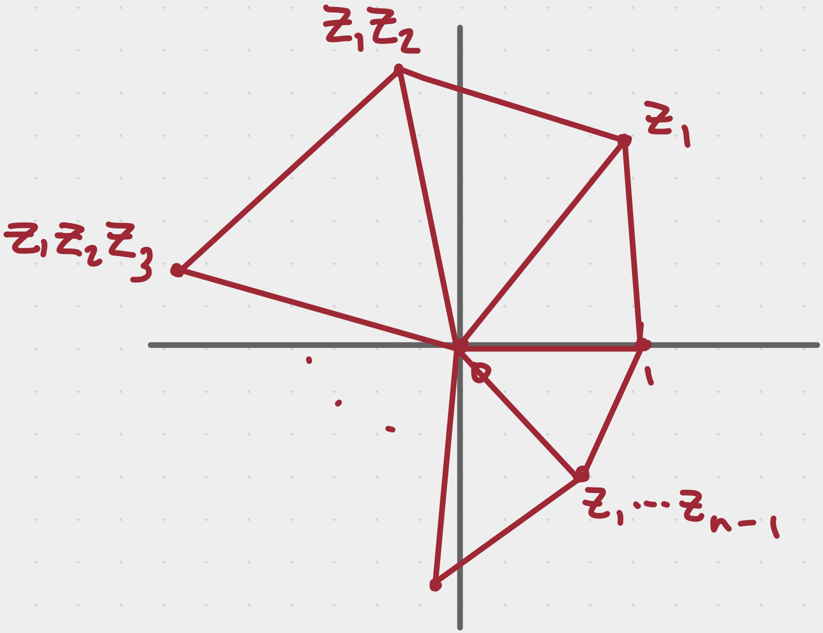
POV ∞

$\mathbb{H}^3$



$\mathbb{C}$

POV ∞

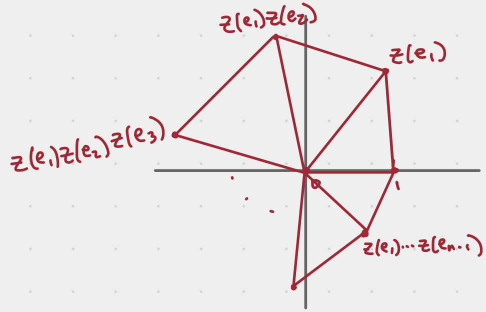


$\mathbb{C}$

Want $z_1 z_2 \cdots z_n = 1$

Gluing Equations: For each edge $e \in T$,

$$\left[\begin{array}{l} \prod_i z(e_i) = 1 \\ \sum_i \arg(z(e_i)) = 2\pi \\ \operatorname{Im} z(e_i) > 0 \end{array}\right]$$

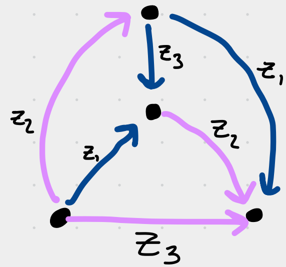


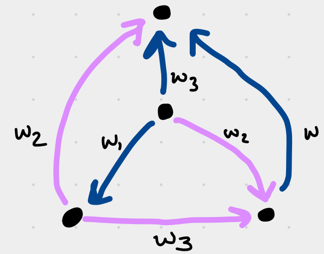
and a vertex condition

If an ideal triangulation T has a solution to the gluing equations, then T is geometric.

$$\left[\text{M has a geometric triangulation}\right] \Longrightarrow \left[\text{M is hyperbolic}\right]$$

$M = S^3 \setminus$ 





$$\boxed{\begin{aligned} z_2 &= \frac{1}{1-z_1} \\ z_3 &= \frac{z_1 - 1}{z_1} \end{aligned}}$$

$$\underbrace{\begin{aligned} z_1^2 z_3 w_1^2 w_3 &= 1 \\ z_2^2 z_3 w_2^2 w_3 &= 1 \end{aligned}}$$

$$z_1 = \frac{1 \pm \sqrt{1 \pm 4/(w_1(w_1-1))}}{2}$$

$$\boxed{z_1 = w_1 = \frac{1+\sqrt{-3}}{2}}$$

[M has a geometric triangulation] $\Longrightarrow$ [M is hyperbolic]

?

Open Question: If M is hyperbolic, does M always have a geometric triangulation?

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Maybe we can understand this question more if we can understand all possible geometric triangulations of a fixed manifold.

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Why is this hard?

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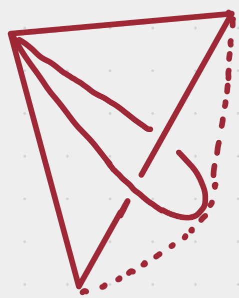
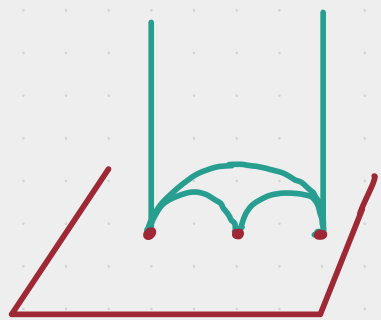




rigid

Open Question: If M is hyperbolic, does M always have a geometric triangulation?

Why is this hard?

The easy things don't work...

Thm (Luo, Schleimer, Tillman; 2007) Geometric triangulations exist virtually.

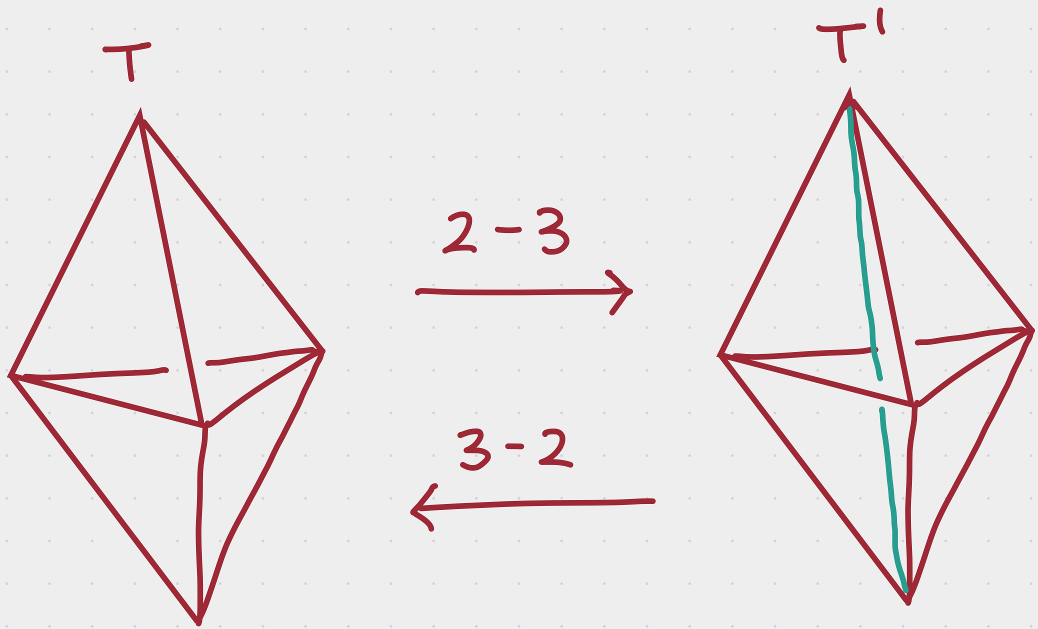
Thm (Futer, Hamilton, Hoffman; 2021) Infinitely many geometric triangulations exist virtually.

Fix a hyperbolic M.

Ambitious Goal: Collect the set of geometric triangulations of M.

If I give you a geometric triangulation of M, can you give me a different geometric triangulation of M?

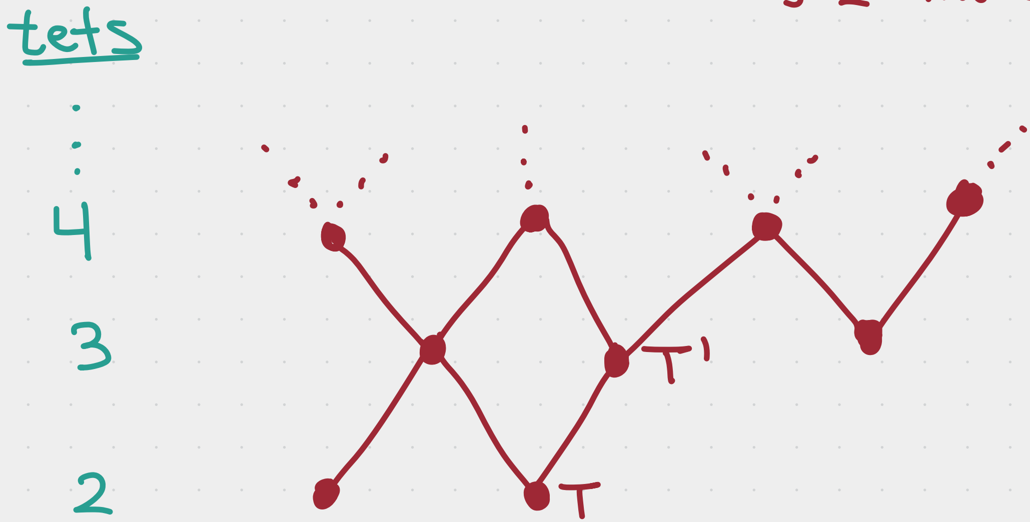
Bistellar Flips



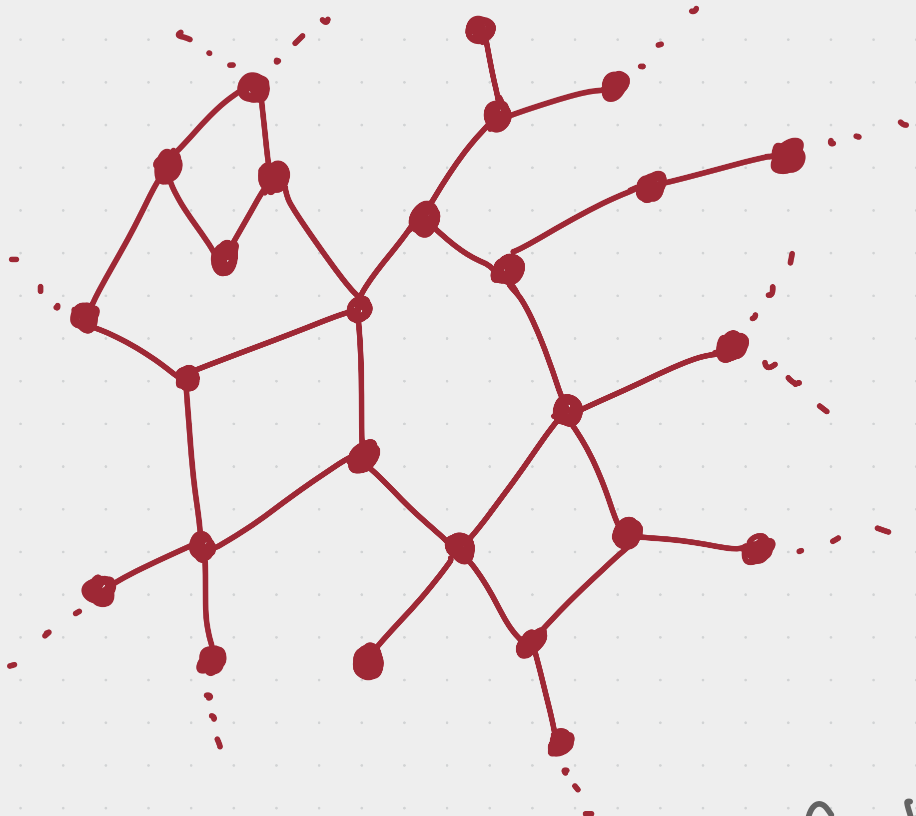
Bistellar Flip Graph

Vertices = { triangulations of M }

Edges = { (T, T') | T and T' differ by a 2-3 or 3-2 move }



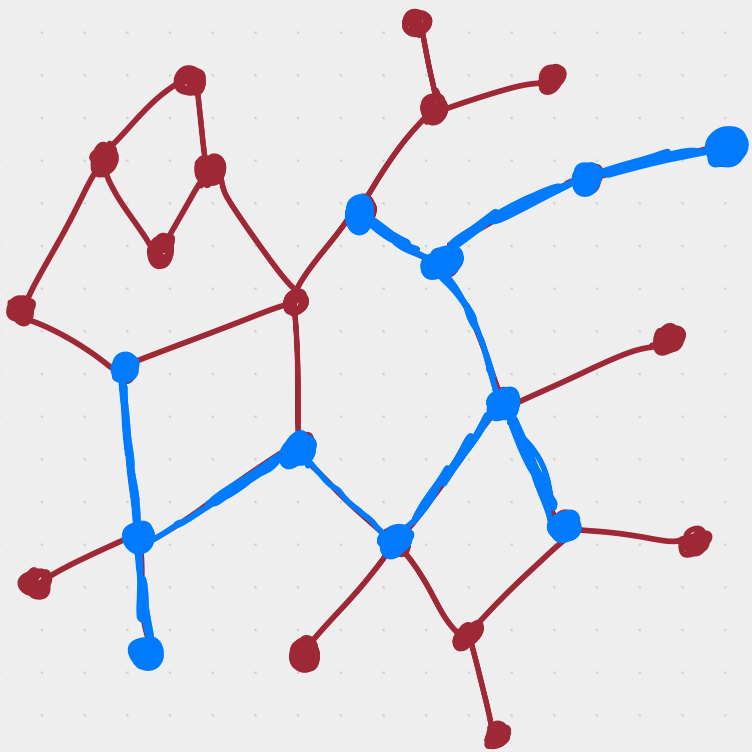
Thm (Amendola; Matveev, Piergallini; Banagl, Friedman; Alexander, Newman, Moise, Pachner) The bistellar flip graph is connected.



Cartoon

Thm (Kalelkar, Schleimer, Segerman; 2024) The subgraph of the bistellar flip graph induced by essential triangulations is connected*.

● = essential (solution to gluing eq's in $\mathbb{C}\setminus\{0,1\}$)

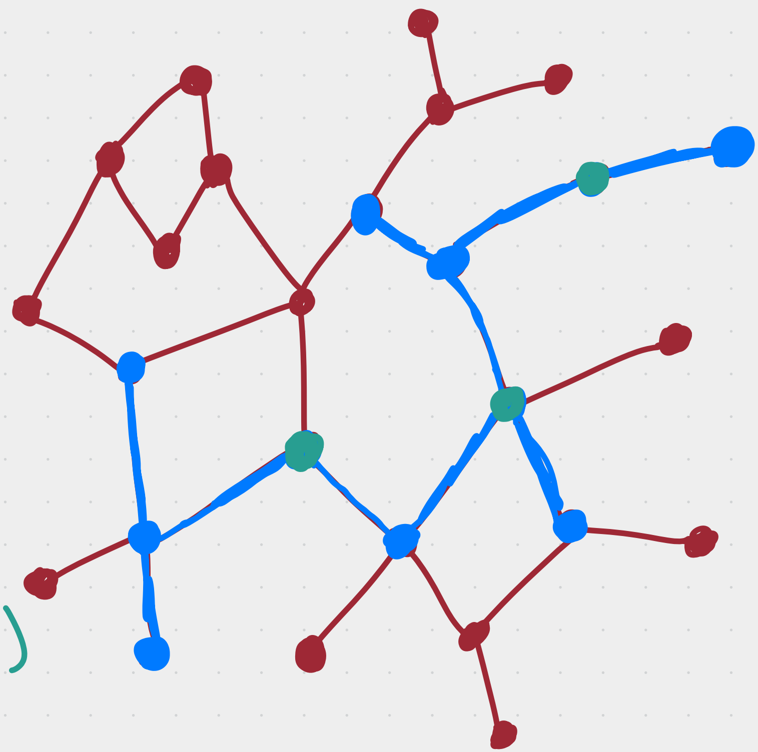


* except for isolated essential triangulations

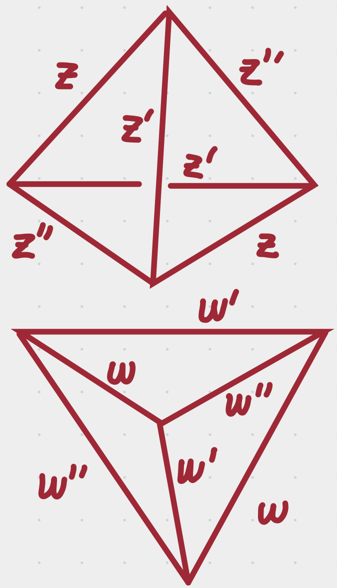
Question Where are the geometric triangulations?
What does the geometric subgraph look like?

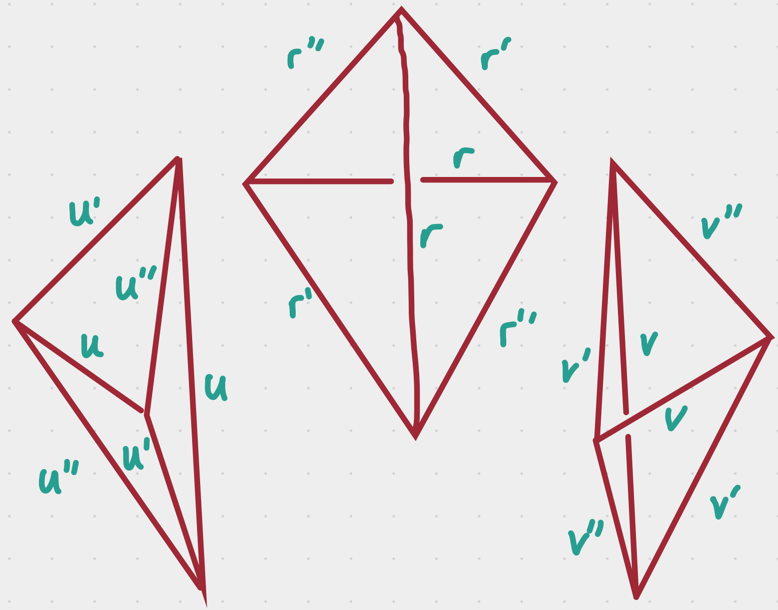
● = essential (solution to gluing eq's in $\mathbb{C}\setminus\{0,1\}$)

● = geometric (solution to gluing eq's in $\{z\in\mathbb{C} \mid \mathrm{Im}(z)>0\}$)



Bistellar Flips in a Geometric Triangulation





$r = z'w'$ $\qquad$ $u = wz''$ $\qquad$ $v = zw''$

Searching

Naive: Start with a geometric triangulation

↓

do a 2-3 move

↓

re-solve gluing equations

Searching

Naive: Start with a geometric triangulation
$\downarrow$
do a 2-3 move
$\downarrow$
re-solve gluing equations

Better: Start with a geometric triangulation
$\downarrow$
do a 2-3 move
$\downarrow$
pass old solutions to new triangulation, and check if they have positive imaginary part

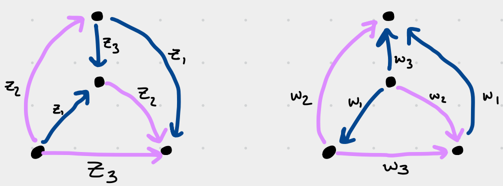
Application

Solving the gluing equations on a triangulation with 1000 tetrahedra sounds hard...

Instead, solve it on a small triangulation and propagate it:

Ex | $S^3 \setminus$ 

cPcbbbiht



$$z_1 = w_1 = \frac{1+\sqrt{-3}}{2}$$

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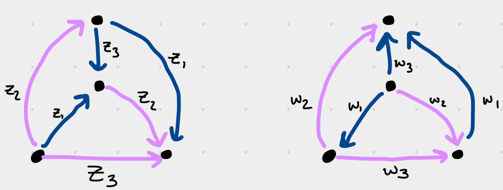

2-3
×998

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$$z_1 = w_1 = \frac{1+\sqrt{-3}}{2}$$

Back to the geometric subgraph:

<u>Question</u> Is the geometric subgraph connected?

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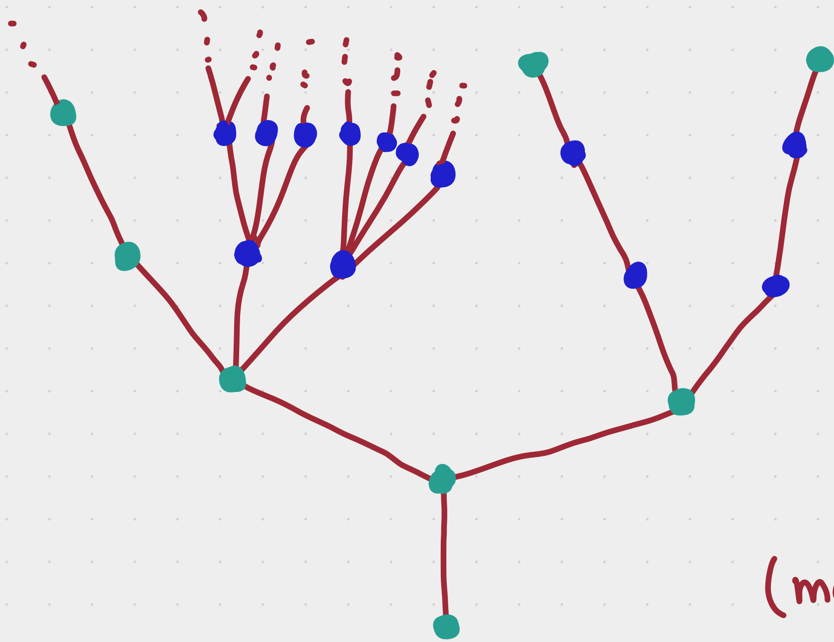
Answer No. (Hoffman, Dadd, Duan, 2015)



$S^3 \setminus$ 

up to 6 tetrahedra

● = geometric

● = has solutions to gluing equations in $\overline{\mathbb{H}}$

(many triangulations not pictured)

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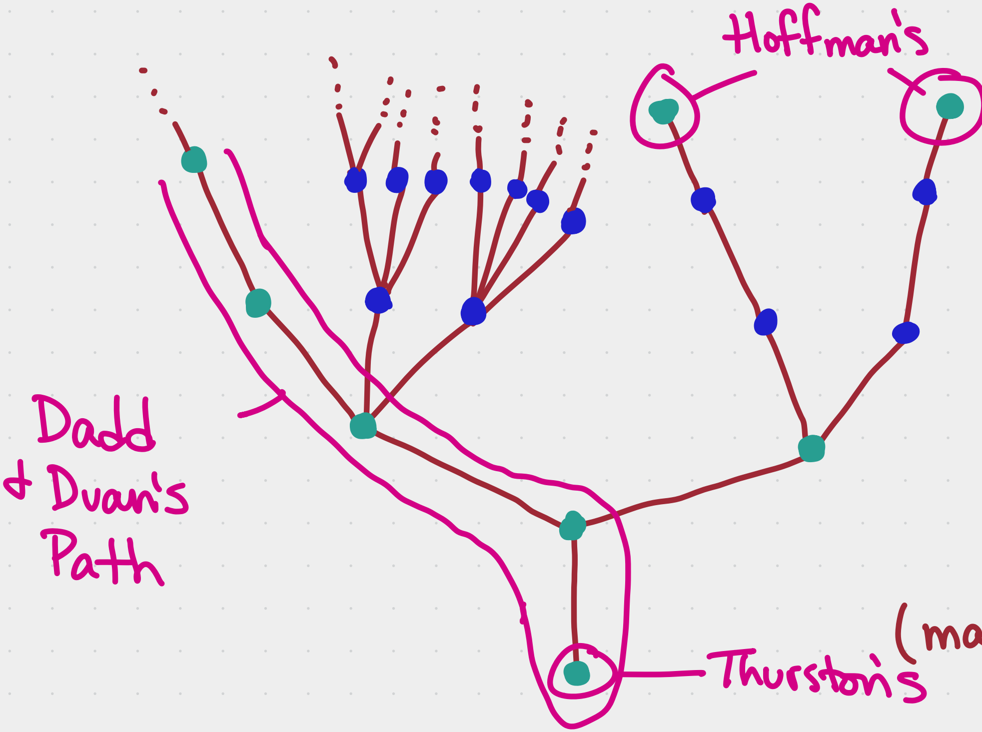

$S^3 \setminus$ 

up to 6 tetrahedra

= geometric

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(many triangulations not pictured)

Question Are there other examples of manifolds with isolated geometric triangulations?

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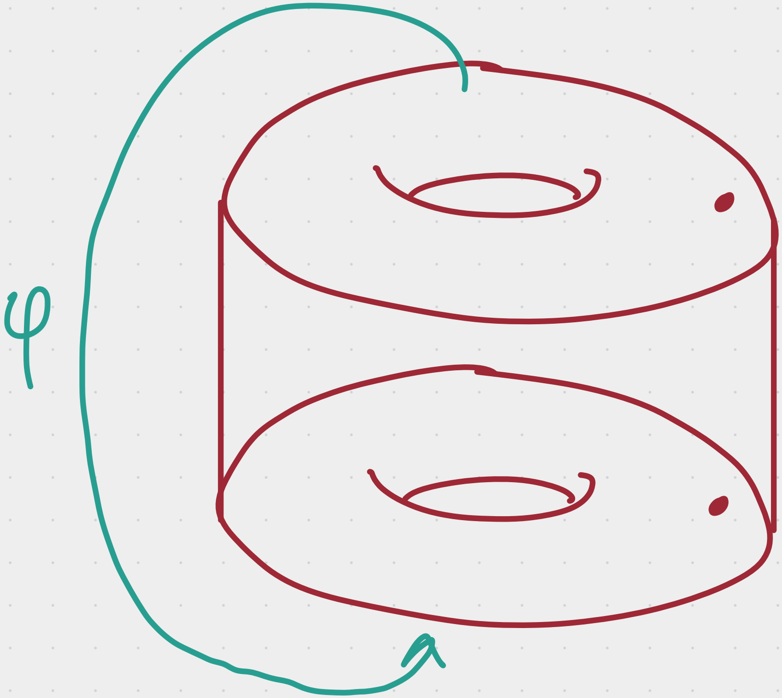
Theorem (B., 2025) There are infinitely many!

The once-punctured torus bundle associated to $\varphi = L^{2m}R^{2n}$ has a minimal triangulation which is an isolated geometric triangulation for all $m, n \in \mathbb{Z}_+$.

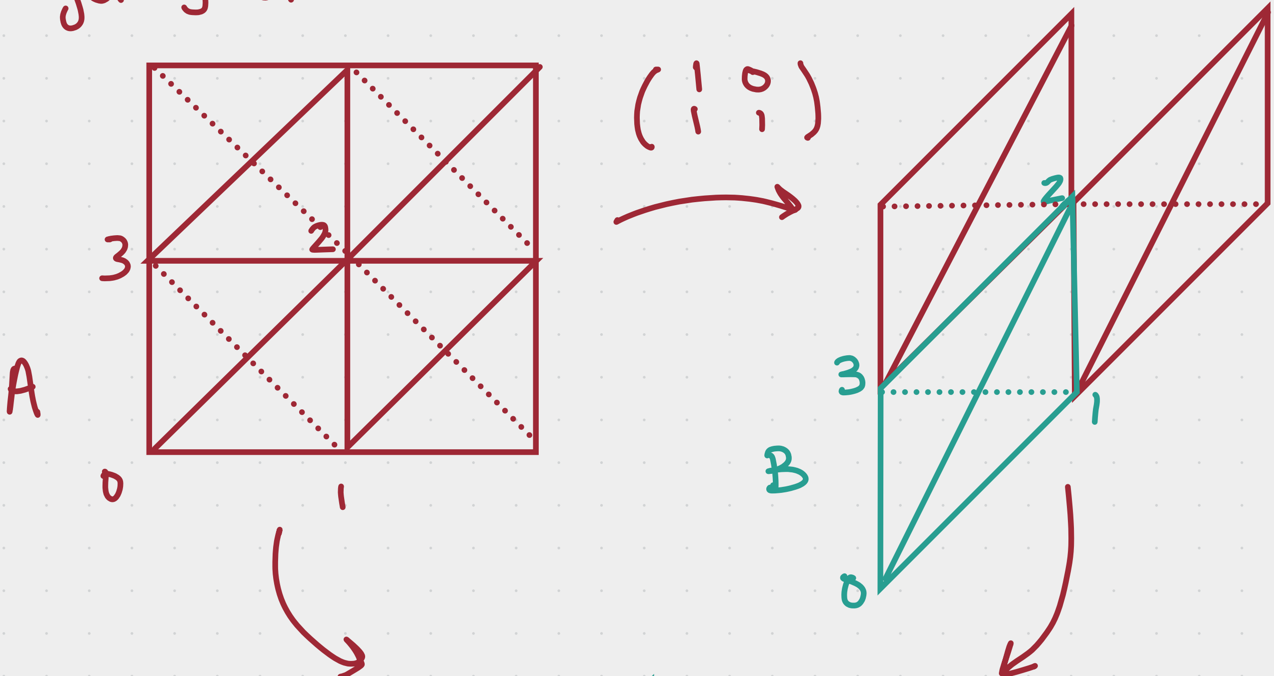
$$\frac{(T^2 \setminus \{0\}) \times I}{(x,0) \sim (\varphi(x),1)}$$

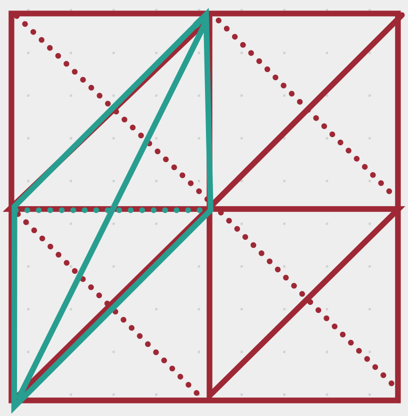


$$\varphi = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{2m} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{2n}$$

$$= L^{2m} R^{2n}$$

"Layering of an L"

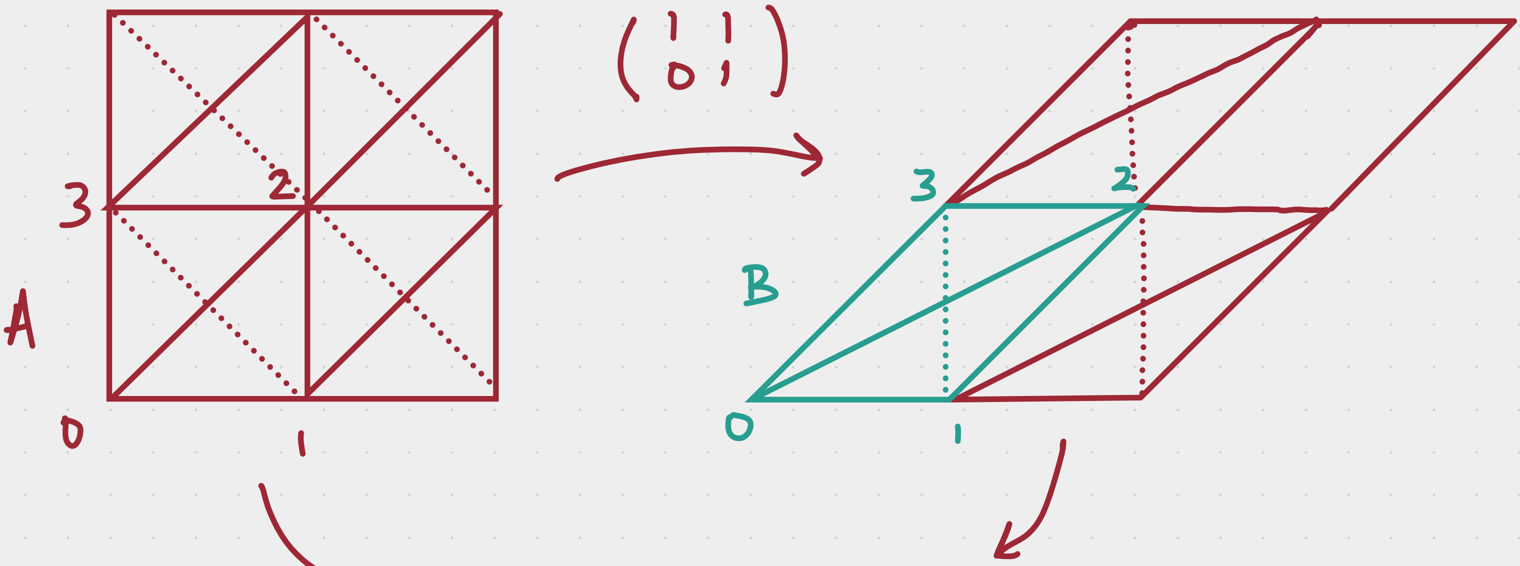




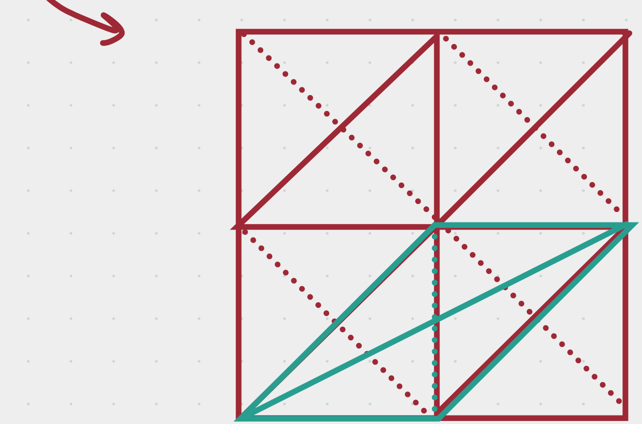
$A(012) = B(312)$

$A(023) = B(013)$

"Layering of an R"



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

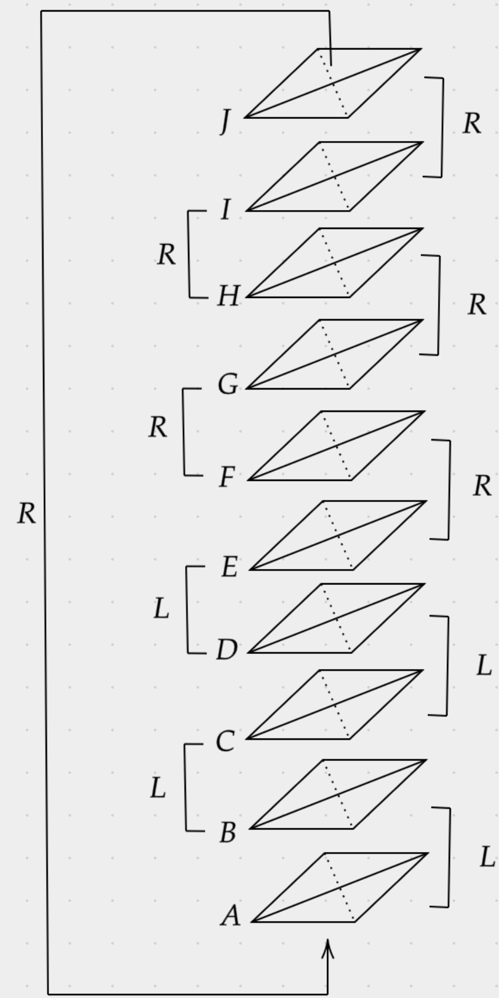


$A(012) = B(013)$

$A(023) = B(123)$

$$M = \frac{(T^2 \setminus \{0\}) \times I}{\sim}$$

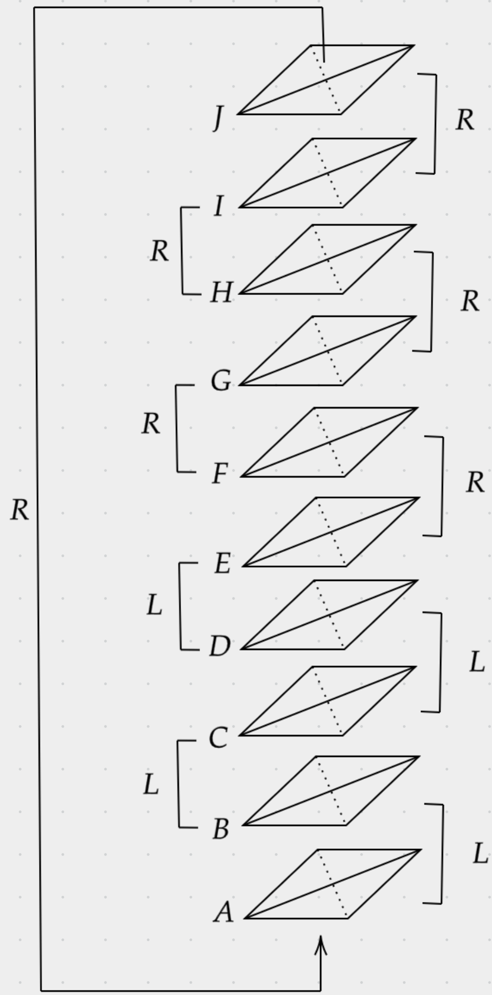
$$\varphi = L^4 R^6$$



$M = (T^2 \setminus \{0\}) \times I \,/\, \sim$

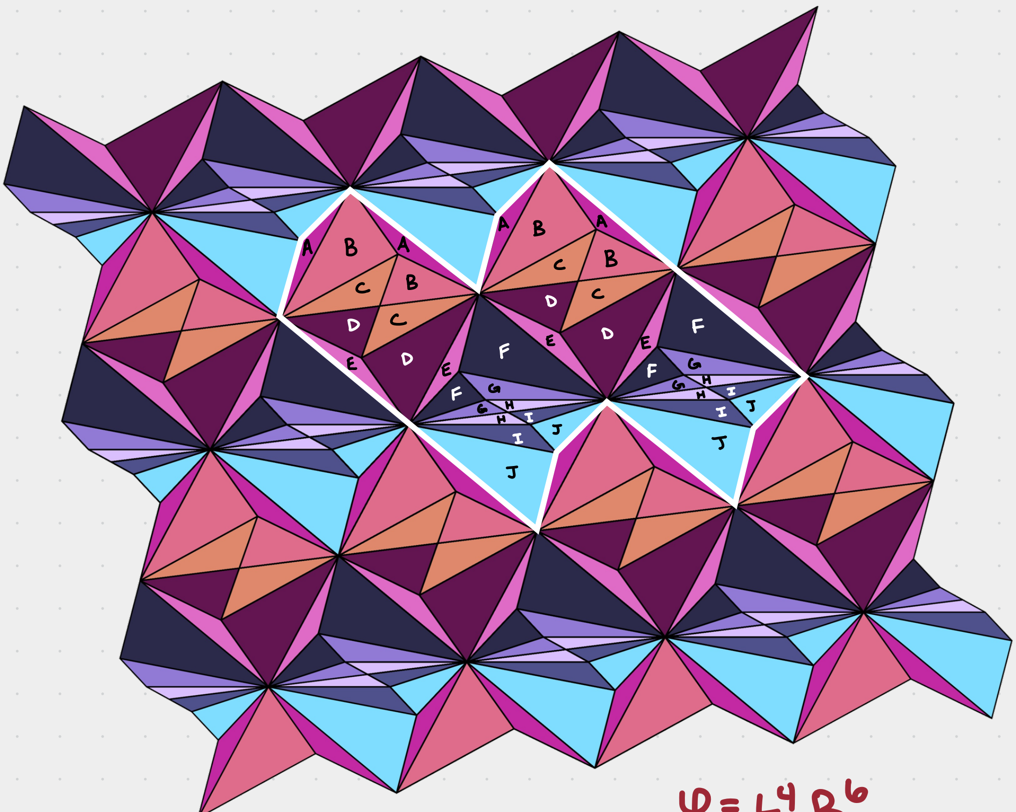
$\varphi = L^4 R^6$

- Construction due to Floyd & Hatcher
- Geometric (Guéritaud, 2006)
- Canonical (Lackenby, 2001)
- Minimal (Jaco, Rubinstein, Spreer, Tillman; 2019)



Strategy:

→ A geometric triangulation of M induces a Euclidean triangulation on $\partial M \cong T^2$.

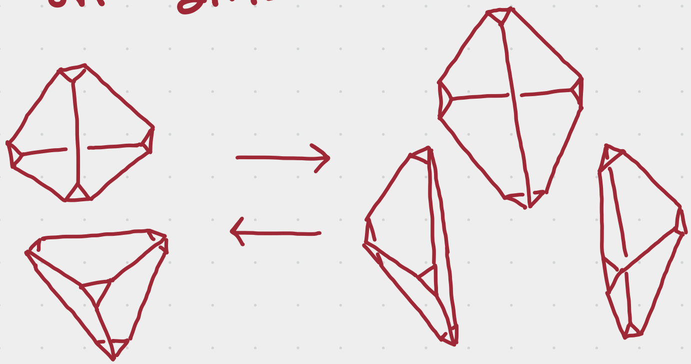


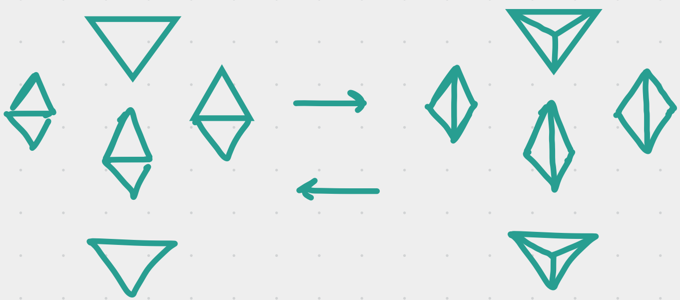
$\varphi = L^4 R^6$

Strategy:

→ A geometric triangulation of M induces a Euclidean triangulation on $\partial M \cong T^2$.

→ A 2-3 move induces 2-2 and 1-3 moves on ∂M.



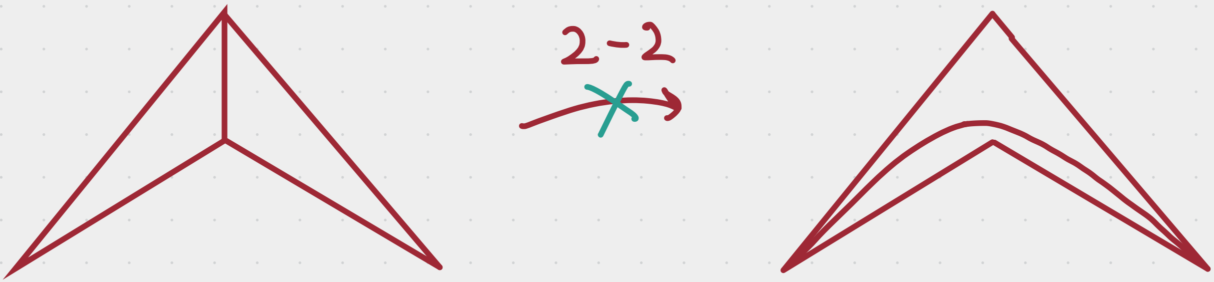


Strategy:

→ A geometric triangulation of M induces a Euclidean triangulation on $\partial M \cong T^2$.

→ A 2-3 move induces 2-2 and 1-3 moves on ∂M.

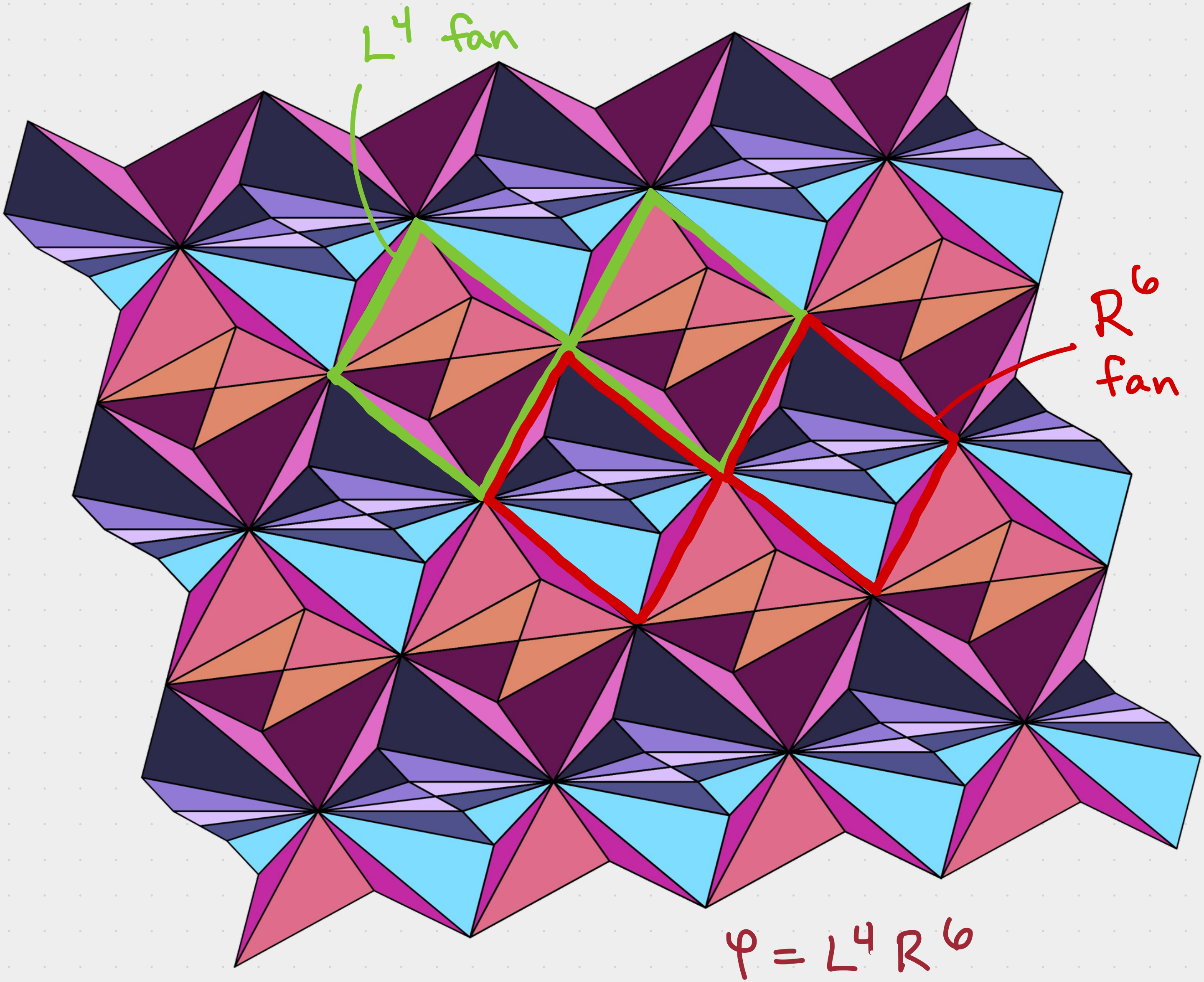
→ An obstruction on the cusp triangulation implies an obstruction in the manifold.



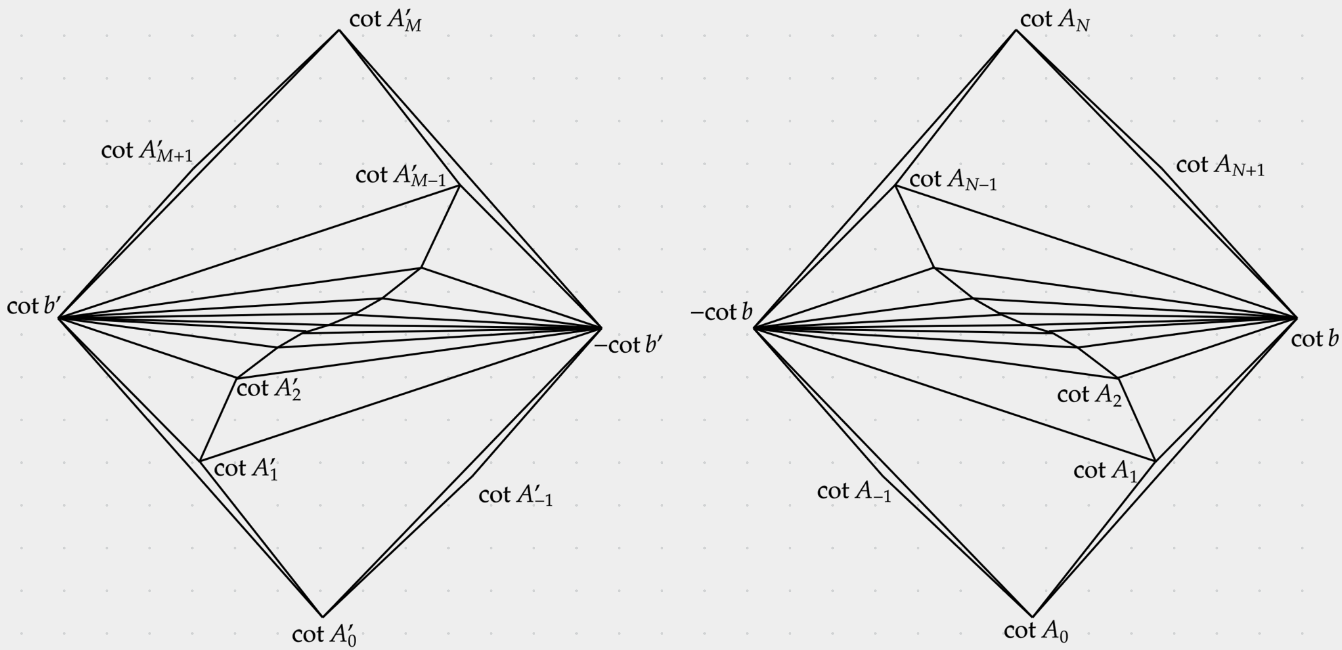
" It's enough to work on the cusp."

Strategy:

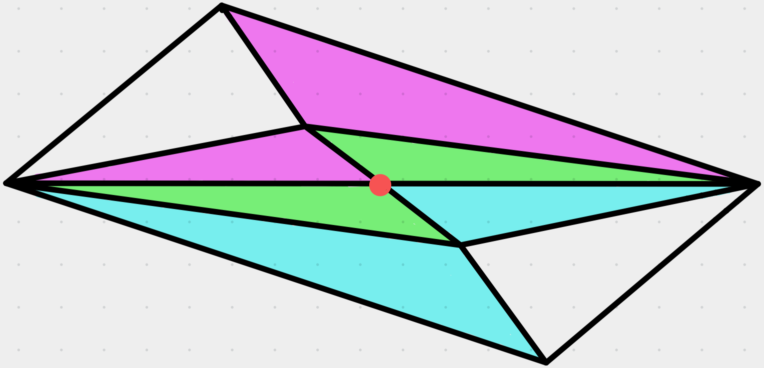
- → A geometric triangulation of M induces a Euclidean triangulation on $\partial M \cong T^2$.
- → A 2-3 move induces 2-2 and 1-3 moves on ∂M.
- → An obstruction on the cusp triangulation implies an obstruction in the manifold.
- → Show there is no disjoint set of three geometric 2-2 moves


L^4 fan
R^6 fan
$\varphi = L^4R^6$

<u>Thm</u> (Guéritaud) For $\varphi = L^m R^n$, vertices of the cusp triangulation inside each fan lie on an embedded graph of $\cot x$.

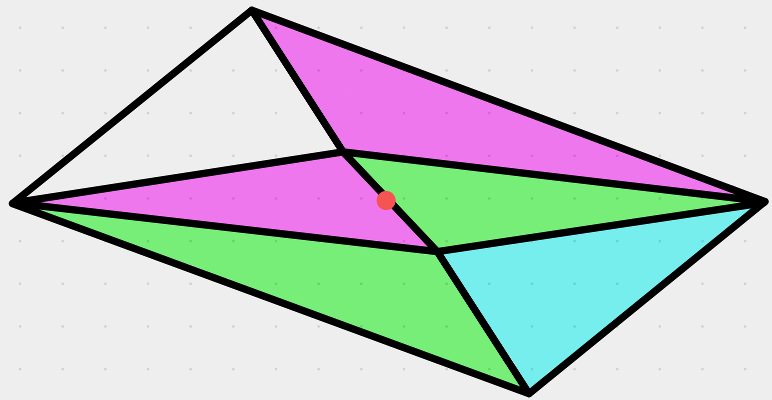


In the fan of L^m

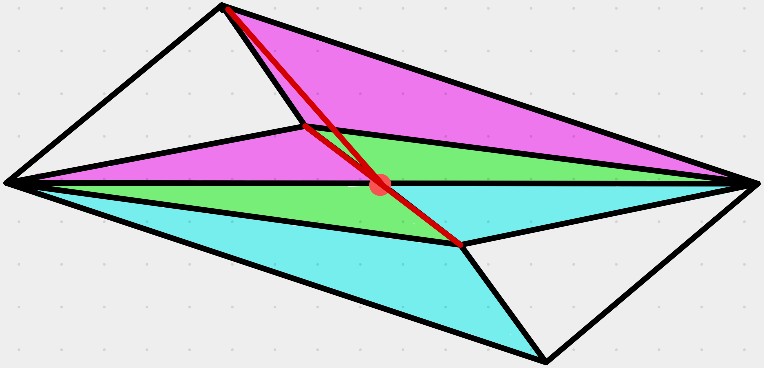


m even

m odd

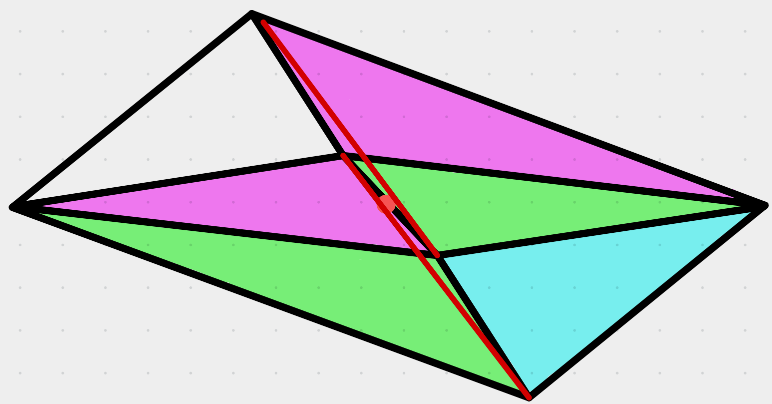


In the fan of L^m



m even

m odd



Thank you !